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Exploring the concept of perfection in 3-hypergraphs

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ABSTRACT

The natural extension of the concept of perfection in graphs to hypergraphs is to define a uniform m-hypergraph, H, as *perfect*, if it satisfies that for every subhypergraph H', $\chi(H') = \lceil \frac{\omega(H')}{m-1} \rceil$, where $\chi(H')$ and $\omega(H')$ are the chromatic and clique number of H', respectively.

It is known that comparability graphs are perfect. In this paper we introduce the concept of comparability 3-hypergraphs (those that can be transitively oriented) with the aim of proving that these are not perfect according to the natural definition. More explicitly, we exhibit three different subfamilies of comparability 3-hypergraphs which show different behaviors in respect to the relationship between the chromatic number and the clique number.

1. Introduction and motivation

In [3] the authors introduce the concepts of orientation for a 3-hypergraph, transitivity for an oriented 3-hypergraph, and define the class of *comparability 3-hypergraphs* as the class of non oriented 3-hypergraphs, which can be transitively oriented (precise definitions are provided in Section 2). These 3-hypergraphs are a natural generalization of (simple) comparability graphs (graphs which can be transitively oriented or, equivalently, graphs associated to a partially ordered set).

Comparability graphs are well known to be perfect graphs. A graph is said to be *perfect* if all of its induced subgraphs have chromatic number equal to their clique number. This concept was introduced by Claude Berge in 1961 [1], we refer the reader to [8] for further details on perfect graphs and highlight that they have been completely characterized in the *Strong Perfect Graph Theorem* [2]. In essence it means that a graph is perfect if for every of its induced subgraphs the chromatic number is as low as possible in terms of its clique number. Thus, it is natural to ask whether or not comparability 3-hypergraphs are perfect in this sense.

The notion of perfection for hypergraphs has already been studied [4,5]. However, the precise concept of perfection for hypergraphs remains imprecise, to the best of our knowledge. Aiming to find a suitable definition of perfection in hypergraphs, we study the relationship between the chromatic number and the clique number of comparability 3-hypergraphs.

We define a 3-*hypergraph* H as usual, $H = (V(H), E(H))$ where $V(H)$ is the set of *vertices* of H, and $E(H) \subseteq \binom{V(H)}{3}$ is the set of *edges*. The *chromatic number*, $\chi(H)$, is defined as the minimum k, such that $V(H)$ can be partitioned into k parts, called *color classes*, in such a way that no edge of H is monochromatic, in other words, no edge is contained in a single

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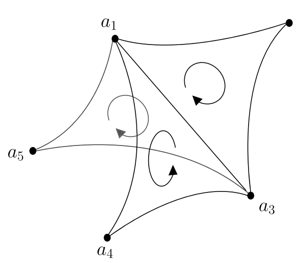

Fig. 1. A non transitive oriented 3-hypergraph for which its underlying graph is a comparability 3-hypergraph.

color class. The *clique number*, $\omega(H)$, of a 3-hypergraph H is the largest cardinality of a subset of $V(H)$ inducing a complete 3-hypergraph.

Given that for any complete 3-hypergraph on n vertices, K_n^3, we have that $\chi(K_n^3) = \lceil \frac{n}{2} \rceil$, then for any 3-hypergraph, H, the following equation holds:

$$\left\lceil \frac{\omega(H)}{2} \right\rceil \leq \chi(H). \tag{1}$$

In this paper we study three important subclasses of comparability 3-hypergraphs which show three different behaviors in relation to (1).

Firstly, we exhibit a family of comparability 3-hypergraphs for which the difference, $\chi(H) - \left\lceil \frac{\omega(H)}{2} \right\rceil$, is arbitrarily large.

Secondly, we exhibit an interesting subclass of comparability 3-hypergraphs, named cyclic permutation 3-hypergraphs (the analogues of permutation graphs), such that their chromatic number is bounded by a (linear) function of its clique number.

Finally, we exhibit another interesting subclass of comparability 3-hypergraphs namely, the ones associated to a family of intervals in the circle. For these hypergraphs the chromatic number is as low as it can be in respect to their clique, that is, equality holds in (1).

The paper is organized as follows: in Section 2 we state the required definitions and preliminary results necessary to prove our main theorems. The main results are stated in Section 3 and the proofs are located in the remaining sections.

2. Definitions and preliminaries

Let X be any set of order n. A *linear ordering* of X is a bijection $\phi : \{1, 2, \ldots, n\} \to X$. A *cyclic ordering* of X is an equivalent class of the set of linear orderings with respect to the *cyclic equivalence relation* defined as: $\phi \sim \psi$, if and only if there exists $k \leq n$, such that $\phi(i) = \psi(i + k)$ for every $i \in \{1, 2, \ldots, n\}$ where $i + k$ is taken modulo n. For the remainder of this paper we will denote each cyclic ordering, $[\phi]$, in cyclic permutation notation, $\big(\phi(1)\,\phi(2)\,\ldots\,\phi(n)\big)$. For example, there are two different cyclic orderings of $\{u, v, w\}$, namely $(u\ v\ w)$ and $(u\ w\ v)$, where $(u\ v\ w) = (v\ w\ u) = (w\ u\ v)$ and $(u\ w\ v) = (v\ u\ w) = (w\ v\ u)$.

Given a 3-hypergraph H, an *orientation* of H is an assignment of exactly one of the two possible cyclic orderings to each of its edges. An orientation of a 3-hypergraph is called an *oriented 3-hypergraph*, and we denote the oriented edges by $O(H)$.

Example 1. Let $H = (V(H), E(H))$ be a 3-hypergraph with $V(H) = \{a_1, a_2, a_3, a_4, a_5\}$ and $E(H) = \{\{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}, \{a_1, a_3, a_5\}\}$, then a possible orientation of H could be $O(H) = \{(a_1\,a_2\,a_3), (a_1\,a_4\,a_3), (a_1\,a_3\,a_5)\}$ obtaining the oriented 3-hypergraph depicted in Fig. 1.

It is usual to associate transitive oriented graphs to partial (linear) orders. Similarly, we can associate to partial cyclic orders (a ternary relation which is cyclic, asymmetric and transitive) transitive oriented 3-hypergraphs in the following manner:

Definition 1. An oriented 3-hypergraph H is said to be *transitive*, if whenever $(u\,v\,z)$ and $(z\,v\,w) \in O(H)$, then $(u\,v\,w) \in O(H)$ (this implies also $(u\,w\,z) \in O(H)$).

Now it is natural to define and study the following class of 3-hypergraphs.

Definition 2. A non-oriented 3-hypergraph is called a *comparability* 3-*hypergraph* if it admits a transitive orientation.[1]

The oriented 3-hypergraph defined in Example 1 is not transitive, however, its underlying 3-hypergraph H is a comparability 3-hypergraph since it can be transitively oriented; take for instance $O'(H) = \{(a_1\,a_3\,a_2), (a_1\,a_3\,a_4), (a_1\,a_3\,a_5)\}$. In contrast, a 3-hypergraph with four vertices and three edges is not a comparability 3-hypergraph.

[1] In [3] the authors defined this class as "cyclic comparability 3-hypergraphs" however we believe that it should be simply called comparability 3-hypergraphs according to the classical concept of comparability graphs.