Discrete Applied Mathematics 210 (2016) 185–194

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Edge intersection graphs of L-shaped paths in grids

Kathie Cameron [a], Steven Chaplick [b,*], Chính T. Hoàng [c]

[a] Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada
[b] Institut für Mathematik, Technische Universität Berlin, Berlin, Germany
[c] Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, ON, Canada

ARTICLE INFO

Article history:
Received 14 January 2014
Received in revised form 5 December 2014
Accepted 29 January 2015
Available online 9 March 2015

Keywords:
Edge intersection graphs
Grid paths
Split graphs
NP-completeness
Recognition algorithms
L-graphs

ABSTRACT

In this paper we continue the study of the edge intersection graphs of one (or zero) bend paths on a rectangular grid. That is, the edge intersection graphs where each vertex is represented by one of the following shapes: ⌞, ⌜, ⌟, ⌝, and we consider zero bend paths (i.e., | and –) to be degenerate ⌞'s. These graphs, called B_1-EPG graphs, were first introduced by Golumbic et al. (2009). We consider the natural subclasses of B_1-EPG formed by the subsets of the four single bend shapes (i.e., {⌞}, {⌞, ⌜}, {⌞, ⌝}, and {⌞, ⌜, ⌝}) and we denote the classes by [⌞], [⌞, ⌜], [⌞, ⌝], and [⌞, ⌜, ⌝] respectively. Note: all other subsets are isomorphic to these up to 90 degree rotation. We show that testing for membership in each of these classes is NP-complete and observe the expected strict inclusions and incomparability (i.e., [⌞] $\subsetneq$ [⌞, ⌜], [⌞, ⌝] $\subsetneq$ [⌞, ⌜, ⌝] $\subsetneq$ B_1-EPG and [⌞, ⌜] is incomparable with [⌞, ⌝]). Additionally, we give characterizations and polytime recognition algorithms for special subclasses of *Split* $\cap$ [⌞].

1. Introduction

A graph G is called an *EPG graph* if G is the intersection graph of paths on a rectilinear grid, where each vertex in G corresponds to a path on the grid and two vertices are adjacent in G if and only if the corresponding paths share an edge on the grid. EPG graphs were introduced by Golumbic et al. [7]. The motivation for studying these graphs comes from circuit layout problems [2]. Golumbic et al. [7] defined a *B_k-EPG graph* to be the edge intersection graph of paths on a grid where the paths are allowed to have at most k bends (turns). The B_0-EPG graphs are exactly the well studied *interval graphs* (the intersection graphs of intervals on a line).

Golumbic and Jamison [6] proved that the recognition problem for the edge intersection graphs of paths in trees (EPT) is NP-complete even when restricted to chordal graphs (i.e., graphs without induced k-cycles for $k \geq 4$). Heldt et al. [8] proved that the recognition problem for B_1-EPG is NP-complete. In a recent paper Epstein et al. [5] have demonstrated that both the coloring problem and the independent set problem are NP-complete on B_1-EPG graphs. They have further shown that these problems can be 4-approximated in polynomial time when a B_1-EPG representation is given and that the clique problem can be solved optimally in polynomial time even without a given EPG representation.

A graph is *chordal* if it does not contain a chordless cycle with at least four vertices as an induced subgraph. A graph is a *split graph* if its vertices can be partitioned into a clique and a stable set; *Split* denotes the class of split graphs. Asinowski and Ries [1] characterized special subclasses of chordal B_1-EPG graphs.

* Corresponding author. Tel.: +49 30 314 28706; fax: +49 30 314 25191.
E-mail addresses: kcameron@wlu.ca (K. Cameron), chaplick@math.tu-berlin.de (S. Chaplick), choang@wlu.ca (C.T. Hoàng).

http://dx.doi.org/10.1016/j.dam.2015.01.039
0166-218X/

Fig. 1. Left: An edge-clique. Right: A claw-clique.

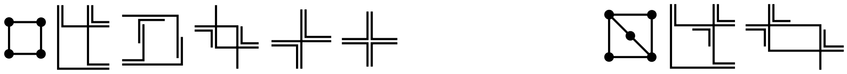

Fig. 2. Left: C_4 and representations of it. Right: $K_{2,3}$ and representations of it.

Consider a B_1-EPG graph G with a path representation on a grid. The paths can be of the following four shapes: $\llcorner$, $\ulcorner$, $\lrcorner$, $\urcorner$. In this paper, we study B_1-EPG graphs whose paths on the grid belong to a proper subset of the four shapes. If $\mathcal{S}$ is a subset of $\{\llcorner, \ulcorner, \lrcorner, \urcorner\}$, then $[\mathcal{S}]$ denotes the class of graphs that can be represented by paths whose shapes belong to $\mathcal{S}$. It is important to note that we also zero-bend paths (i.e., vertical and horizontal line segments) to be $\llcorner$s. In particular, an $[\llcorner]$-representation of a graph may have some of its vertices represented as zero-bend paths. We are especially interested in the class $[\llcorner]$ of B_1-EPG graphs whose paths are of the type $\llcorner$. Our main results are:

- Establishment of expected separation between the classes: $[\llcorner] \subsetneq [\llcorner, \ulcorner], [\llcorner, \urcorner] \subsetneq [\llcorner, \ulcorner, \urcorner] \subsetneq B_1$-EPG and the incomparability between $[\llcorner, \ulcorner]$ and $[\llcorner, \urcorner]$.
- A proof of NP-completeness of recognition of $[\llcorner]$ and of each of the other subclasses of B_1-EPG mentioned above.
- Characterizations of, and recognition algorithms for gem-free split $[\llcorner]$-graphs and bull-free split $[\llcorner]$-graphs.

In Section 2, we discuss background results and establish some properties of B_1-EPG graphs. In Section 3, we show that recognition of $[\llcorner]$ and of each of the other subclasses is an NP-complete problem. In Section 4, we give polytime recognition algorithms for the classes of gem-free split $[\llcorner]$-graphs and of bull-free split $[\llcorner]$-graphs. We conclude with some open questions in Section 5. An extended abstract of this paper appeared at LAGOS'13 Cameron et al. (2013) [4].

2. Properties of B_1-EPG graphs

Let $\mathcal{P}$ be a collection of nontrivial simple paths on a rectilinear grid $\mathcal{G}$. (The end-points of each path are grid points.) The *edge intersection graph* $EPG(\mathcal{P})$ has a vertex v for each path $P_v \in \mathcal{P}$ and two vertices are adjacent in $EPG(\mathcal{P})$ if the corresponding paths in $\mathcal{P}$ share an edge of $\mathcal{G}$. For any grid edge e, the set of paths containing e is a clique in $EPG(\mathcal{P})$; such a clique is called an *edge-clique* [7]. A *claw* in a grid consists of three grid edges meeting at a grid point. The set of paths which contain two of the three edges of a claw is a clique; such a clique is called a *claw-clique* [7] (see Fig. 1).

Lemma 1 ([7]). *Consider a B_1-EPG representation of a graph G. Every clique in G corresponds to either an edge-clique or a claw-clique.*

The *neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x. A set of vertices is *stable* if no two are adjacent. An *asteroidal triple* (AT) is a stable set of size three such that for every pair, there is a path between them which avoids the neighborhood of the other vertex.

Lemma 2 (*AT Lemma [1, Theorem 9]*). *In a B_1-EPG graph, no vertex can have an AT in its neighborhood.*

Let C_4 denote the chordless cycle a, b, c, d, a on four vertices. Golumbic et al. [7] proved that any B_1-EPG representation of C_4 corresponds to what they call a "true pie", a "false pie", or a "frame". True and false pies require paths other than $\llcorner$'s. A *frame* is a rectangle in the grid $\mathcal{G}$ such that each corner is the bend-point for one of P_a, P_b, P_c and P_d; $P_a \cap P_b$, $P_b \cap P_c$, $P_c \cap P_d$, and $P_d \cap P_a$ each contain at least one grid edge; and $P_a \cap P_c$ and $P_b \cap P_d$ each do not contain an grid edge. Consider the C_4 and four representations of it shown in Fig. 2. The first three representations are frames, the fourth is a false pie, and the fifth is a true pie. It follows that:

Lemma 3 (*C_4 Lemma*). *In an $[\llcorner]$- or $[\llcorner, \ulcorner]$-representation of a C_4, every $\llcorner$, and $\ulcorner$ has a neighbor on both its vertical segment and on its horizontal segment.*

Observation 4. *$K_{2,3}$ is in $[\llcorner, \urcorner]$.*

Proof. See Fig. 2 for an $[\llcorner, \urcorner]$-representation of $K_{2,3}$. □

Lemma 5 (*$K_{2,3}$ Lemma*). *In an $[\llcorner, \urcorner]$-representation of a $K_{2,3}$ every $\llcorner$, and $\urcorner$ has a neighbor on both its vertical segment and on its horizontal segment.*

Proof. Consider $K_{2,3}$ to be the complete bipartite graph with bipartition $\{\{a, b\}, \{c, d, e\}\}$. Note that each of the following is a C_4: a, c, b, d, a; a, c, b, e, a; and a, d, b, e, a. As noted above, any B_1-EPG representation of C_4 corresponds to a "true pie", a "false pie", or a "frame". True pies require paths of all four types, but false pies and frames can be made from just $\llcorner$'s and $\urcorner$'s.