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# Null and non-rainbow colorings of projective plane and sphere triangulations

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ABSTRACT

By considering graphs as topological spaces we introduce, at the level of homology, the notion of a null coloring, which provides new information on the task of clarifying the structure of cycles in a graph. We prove that for any graph $G$ a maximal null coloring $f$ is such that the quotient graph $G/f$ is acyclic. As an application, for maximal planar graphs (sphere triangulations) of order $n \geq 4$, we prove that a vertex-coloring containing no rainbow faces uses at most $\left\lfloor \frac{2n-1}{3} \right\rfloor$ colors, and this is best possible. For maximal graphs embedded on the projective plane we obtain the analogous best bound $\left\lfloor \frac{2n+1}{3} \right\rfloor$.


## 1. Introduction

In this work we consider graphs as topological spaces. A *simple graph*, which combinatorially is a pair of sets $G = (V, E)$ (where $V$ is a finite set of elements called *vertices*, and $E$ is a set of 2-element subsets of $V$ with elements called *edges*) will be regarded as a subset of points in $\mathbb{R}^3$. In this framework we denote by $H_1(G)$ the *first homology group* of $G$, which is a free abelian group with $m - n + 1$ generators where $n$ and $m$ are respectively the number of vertices and edges of the graph. Roughly speaking, the first homology group $H_1(G)$ of a graph $G$ measures the number of independent 1-dimensional holes (cycles) in the graph.

In this work we are interested in connected graphs. In the non connected case we can proceed by considering each component independently.

For simple connected graphs we consider vertex colorings, which are not necessarily proper colorings. It is well known that a $k$-coloring of $G$ can be seen as a homomorphism $f : G \to K_k$, and thus, in our context, as a continuous mapping from $G$ to $K_k$. Hence, every $k$-coloring $f$ induces a group homomorphism $f_* : H_1(G) \to H_1(K_k)$. At this level we introduce a new type of coloring: we say that a vertex $k$-coloring $f$ of $G$ is a *null coloring*, if $f_*$ is zero, that is, all the cycles of $G$ are mapped into closed trivial walks in $K_k$. Since for any graph $G$ a trivial example of a null coloring is to color all its vertices with one single color, then the interest is on finding null colorings with the maximum number of colors. In Section 2 we give precise definitions. In Section 3 we prove our main theorem, Theorem 1, on maximal null colorings.

Finally, in Section 4 we provide as a corollary of Theorem 1, the exact value from which tricolored faces are inevitable in any coloring of a given sphere triangulation, respectively in any coloring of a given projective plane triangulation. The problem of maximizing the number of colors in a vertex coloring avoiding rainbow faces has been studied recently in several papers [3,5,6,8,9]. In Section 4, before stating and proving our result, we summarize some known results in the area.

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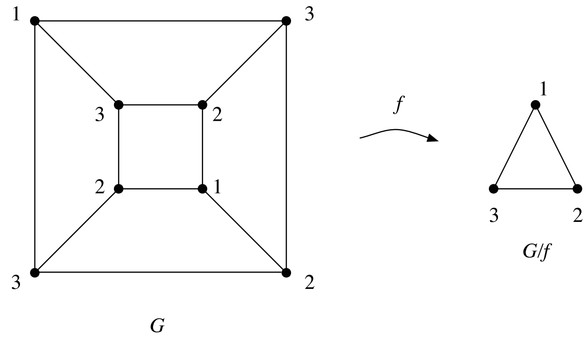


**Fig. 1.** A null coloring $f$ such that $G/f$ is not a tree.

## 2. Null colorings

Let $G$ be a simple connected graph. A *realization* of $G$ is a set of points in a real vector space $\mathbb{R}^N$, where each vertex is a different point, and each edge is a straight segment joining its corresponding pair of vertices, where no vertex lies on an edge, except at one of its end-points, and two edges can meet only in a common end-point. It is well known that not every graph can be realized in $\mathbb{R}^2$, but all graphs can be realized in $\mathbb{R}^3$. In this manner we regard a graph $G$ as a topological space: a subset of points in $\mathbb{R}^3$ corresponding to a realization of $G$.

Let $G$ be a simple connected graph with $|V(G)| = n$ and $|E(G)| = m$. Consider a realization $G \subset \mathbb{R}^3$ and denote by $H_1(G)$ the *first homology group* of $G$, which is a free abelian group with $m - n + 1$ generators. The first homology group, $H_1(G)$, is isomorphic to the group of 1-cycles in $G$. For instance, if $T$ is a tree, then $H_1(T) \simeq 0$, and for any cycle $C_n$ we have $H_1(C_n) \simeq \mathbb{Z}$. In fact, a basis $\beta$ of generators for $H_1(G)$ is given by choosing a spanning tree $T$ of $G$, and an orientation for the edges $\{e_1, \ldots, e_{m-n+1}\}$ of $G-T$. So, giving a closed walk $W$ in $G$, it represents the element $\alpha_1 \oplus \cdots \oplus \alpha_{m-n+1} \in \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} = H_1(G)$, according with the basis $\beta$, where $\alpha_i$ is the direct sum of how many times the direct edge $e_i$ is transversed by $W$ (for further information on this topic, see [4] and references therein).

A *k-coloring* of $G$ is a surjective mapping $f : V(G) \to \{1, 2, \ldots, k\}$, or equally a partition of $V(G)$ into exactly $k$ nonempty parts called *color classes*. It is well known that a $k$-coloring of a graph $G$ can be viewed as a homomorphism $f : G \to K_k$ (note that since we allow colorings which are not proper the homomorphism may be reflexive). By considering graphs as topological spaces we can think on a $k$-coloring of $G$ as a continuous mapping $f : G \to K_k$ by sending each vertex of $G$ to its image in $K_k$ and extending the map linearly to the edges. Thus, at the level of homology, a $k$-coloring $f$ induces a group homomorphism $f_* : H_1(G) \to H_1(K_k)$. Such a group homomorphism is called *zero*, if for every $c \in H_1(G)$, then $f_*(c) = 0 \in H_1(K_k)$.

**Definition 1.** For a simple connected graph $G$, a $k$-coloring $f : G \to K_k$ is called a *null coloring*, if and only if, $f_* : H_1(G) \to H_1(K_k)$ is zero.

Let $f$ be a coloring of $G$, we define the *quotient graph* $G/f$ as the graph with vertices being the color classes, and two color classes are adjacent, if there is an edge in $G$ which incident vertices have those colors.

Naturally, if a coloring $f$ of a connected graph $G$ satisfies that $G/f$ is a tree, then $f$ is null. The opposite is not true as shown in the example illustrated in Fig. 1 (the coloring depicted is null since every cycle in $G$ is mapped into a close trivial walk in $G/f$, that is, a walk that topologically can be reduced to a point in $G/f$).

## 3. Maximal null colorings

Let $G$ be a connected graph and $f$ be a $k$-coloring of $G$. We called $f$ *a maximal null coloring* if $f$ is a null coloring and there are no null $(k+1)$-colorings of $G$.

As shown in Fig. 1, not every null coloring $f$ satisfies that the quotient graph $G/f$ is acyclic. Nevertheless, this is true for maximal null colorings. That is, a maximal null coloring satisfies that its quotient graph is acyclic as we will prove next.

**Theorem 1.** *Let $G$ be a simple connected graph, and $f$ be a maximal null coloring of $G$. Then $G/f$ is a tree.*

We first prove some lemmas. Let $G$ be a simple connected graph and $f$ be a coloring of $G$. Let $u, v \in V(G)$ be two vertices with the same color, that is $f(u) = f(v)$. Denote by $G'$ the graph which is obtained from $G$ by identifying the vertices $u$ and $v$, and denote this vertex of $G'$ by $uv$. Let $h : G \to G'$ be the corresponding homomorphism. The coloring $f$ of $G$ naturally induces a coloring $f'$ of $G'$ by defining $f'(uv) = f(u) = f(v)$ and $f'(x) = f(x)$ if $x \notin \{u, v\}$. Obviously $G/f = G'/f'$. We denote by $d(u, v)$ the distance between $u$ and $v$ in $G$.

**Lemma 1.** *For a coloring $f$ of $G$, and two vertices $u, v \in V(G)$ with the same color, let $G'$, $f'$, and $h : G \to G'$ as defined above. If $d(u, v) \leq 2$, then $h_* : H_1(G) \to H_1(G')$ is an epimorphism.*