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On Ramsey numbers of complete graphs with dropped stars

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ABSTRACT

Let $r(G, H)$ be the smallest integer N such that for any 2-coloring (say, red and blue) of the edges of K_n, $n \geqslant N$, there is either a red copy of G or a blue copy of H. Let $K_n - K_{1,s}$ be the complete graph on n vertices from which the edges of $K_{1,s}$ are dropped. In this note we present exact values for $r(K_m - K_{1,1}, K_n - K_{1,s})$ and new upper bounds for $r(K_m, K_n - K_{1,s})$ in numerous cases. We also present some results for the Ramsey number of Wheels versus $K_n - K_{1,s}$.

1. Introduction

Let G and H be two graphs. Let $r(G, H)$ be the smallest integer N such that for any 2-coloring (say, red and blue) of the edges of K_n, $n \geqslant N$ there is either a red copy of G or a blue copy of H. Let $K_n - K_{1,s}$ be the complete graph on n vertices from which the edges of $K_{1,s}$ are dropped. We notice that $K_n - K_{1,1} = K_n - e$ (the complete graph on n vertices from which an edge is dropped) and $K_n - K_{1,2} = K_n - P_3$ (the complete graph on n vertices from which a path on three vertices is dropped).

In this note we investigate $r(K_m - e, K_n - K_{1,s})$ and $r(K_m, K_n - K_{1,s})$ for a variety of integers m, n and s. In the next section, we prove our main result (Theorem 1). In Section 3, we will present exact values for $r(K_m - e, K_n - K_{1,s})$ when $n = 3$ or 4 and some values of m and s. In Section 4, new upper bounds for $r(K_m, K_n - P_3)$ for several integers m and n are given. In Section 5, we give new upper bounds for $r(K_m, K_n - K_{1,s})$ when $m, s \geqslant 3$ and several values of n. In Section 6, we present some equalities for $r(K_4, K_n - K_{1,s})$ extending the validity of some results given in [3]. Finally, in Section 7, we will present results concerning the Ramsey number of the Wheel W_5 versus $K_n - K_{1,s}$. We present exact values for $r(W_5, K_6 - K_{1,s})$ when $s = 3$ and 4 and the equalities $r(W_5, K_n - K_{1,s}) = r(W_5, K_{n-1})$ when $n = 7$ and 8 for some values of s.

Some known values/bounds for specific $r(K_m, K_n)$ needed for this paper are given in Appendix.

2. Main result

Let G be a graph and denote by G^v the graph obtained from G to which a new vertex v, incident to all the vertices of G, is added. Our main result is the following

Theorem 1. *Let n and s be positive integers. Let G_1 be any graph and let N be an integer such that $N \geqslant r(G_1^v, K_n)$. If $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant r(G_1, K_{n+1} - K_{1,s})$ then $r(G_1^v, K_{n+1} - K_{1,s}) \leqslant N$.*

Proof. Let K_N be a complete graph on N vertices and consider any 2-coloring of the edges of K_N (say, red and blue). We shall show that there is either a G_1^v red or a $K_{n+1} - K_{1,s}$ blue. Since $N \geqslant r(G_1^v, K_n)$ then K_N has a red G_1^v or a blue K_n. In the former case we are done, so let us suppose that K_N admit a blue K_n, that we will denote by H. We have two cases.

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Case (1) There exists a vertex $u \in V(K_N \setminus H)$ such that $|N_H^r(u)| \leqslant s$ where $N_H^r(u)$ is the set of vertices in H that are joined to u by a red edge. In this case, we may construct the blue graph $G' = K_{n+1} - K_{1,|N_H^r(u)|}$, this is done by taking H (containing n vertices) and vertex u together with the blue edges between u and the vertices of H. Now, since $|N_H^r(u)| \leqslant s$ then the graph $K_{n+1} - K_{1,s}$ is contained in G' (and thus we found a blue $K_{n+1} - K_{1,s}$).
Case (2) $|N_H^r(u)| > s$ for every vertex $u \in V(K_N \setminus H)$. Then we have that the number of red edges $\{x, y\}$ with $x \in V(H)$ and $y \in V(K_N \setminus H)$ is at least $(N - n)(s + 1)$. So, by the pigeon hole principle, we have that there exists at least one vertex $v \in V(H)$ such that $d^r_{K_N \setminus H}(v) \geqslant \left\lceil \frac{(s+1)(N-n)}{n} \right\rceil$, where $d^r_{K_N \setminus H}(v) = \left| N^r_{K_N \setminus H}(v) \right|$ and $N^r_{K_N \setminus H}(v)$ denotes the set of vertices in $K_N \setminus H$ incident to v with a red edge. But since $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant r(G_1, K_{n+1} - K_{1,s})$ then the graph induced by $N^r_{K_N \setminus H}(v)$ has either a blue $K_{n+1} - K_{1,s}$ (and we are done) or a red G_1 to which we add vertex v to find a red G^v as desired. □

3. Some exact values for $r(K_m - e, K_n - K_{1,s})$

Let $s \geqslant 1$ be an integer. We clearly have that

$$r(K_3 - e, K_m) \leqslant r(K_3 - e, K_{m+1} - K_{1,s}).$$

Since

$$r(K_3 - e, K_{m+1} - K_{1,s}) \leqslant r(K_3 - e, K_{m+1} - e)$$

and (see [10])

$$r(K_3 - e, K_m) = r(K_3 - e, K_{m+1} - e) = 2m - 1$$

then

$$r(K_3 - e, K_{m+1} - K_{1,s}) = 2m - 1 \quad \text{for each } s = 1, \ldots, m - 1.$$

3.1. Case $m = 4$

Corollary 1. (a) $r(K_4 - e, K_5 - K_{1,3}) = 11$.
(b) $r(K_4 - e, K_6 - K_{1,s}) = 16$ *for any* $3 \leqslant s \leqslant 4$.
(c) $r(K_4 - e, K_7 - K_{1,s}) = 21$ *for any* $4 \leqslant s \leqslant 5$.

Proof. (a) It is clear that $r(K_4 - e, K_4) \leqslant r(K_4 - e, K_5 - K_{1,3})$. Since $r(K_4 - e, K_4) = 11$ (see [10]) then $11 \leqslant r(K_4 - e, K_5 - K_{1,3})$. We will now show that $r(K_4 - e, K_5 - K_{1,3}) \leqslant 11$. By taking $N = 11$, $s = 3$ and $n = 4$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4\times 7}{4} \right\rceil = 7 = r(K_3 - e, K_5 - K_{1,3})$ and so, by Theorem 1, we have $r(K_4 - e, K_5 - K_{1,3}) \leqslant 11$, and the result follows.

The proofs for (b) and (c) are analogues. We just need to check that conditions of Theorem 1 are satisfied by taking: $N = r(K_4 - e, K_5) = 16$ for (b) and $N = r(K_4 - e, K_6) = 21$ for (c). □

We notice that Corollary 1(a) is claimed in [8] without a proof. Corollary 1(b) can also be obtained by using that $r(K_4 - e, K_6 - P_3) = 16$ [9] since $16 = r(K_4 - e, K_6 - P_3) \geqslant r(K_4 - e, K_6 - K_{1,s}) \geqslant r(K_4 - e, K_5) = 16$ for $s \in \{3, 4\}$. Corollary 1(c) was first posed by Hoeth and Mengersen [9]. The best known upper bounds for $r(K_4 - e, K_7 - K_{1,3})$ and $r(K_4 - e, K_7 - P_3)$ are obtained by applying the following classical recursive formula:

$$r(K_m - e, K_n - K_{1,s}) \leqslant r(K_{m-1} - e, K_n - K_{1,s}) + r(K_m - e, K_{n-1} - K_{1,s}). \quad (1)$$

Hence

$$r(K_4 - e, K_7 - K_{1,3}) \leqslant r(K_3 - e, K_7 - K_{1,3}) + r(K_4 - e, K_6 - K_{1,3}) = 11 + 16 = 27$$

and

$$r(K_4 - e, K_7 - P_3) \leqslant r(K_3 - e, K_7 - P_3) + r(K_4 - e, K_6 - P_3) = 11 + 16 = 27.$$

We are able to improve the above upper bounds.

Corollary 2. $21 \leqslant r(K_4 - e, K_7 - K_{1,3}) \leqslant 22$.

Proof. It is clear that $r(K_4 - e, K_6) \leqslant r(K_4 - e, K_7 - K_{1,3})$. Since $r(K_4 - e, K_6) = 21$ (see [10]), then $21 \leqslant r(K_4 - e, K_7 - K_{1,3})$. We will now show that $r(K_4 - e, K_7 - K_{1,3}) \leqslant 22$. By taking $N = 22$, $s = 3$ and $n = 6$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4\times 16}{6} \right\rceil = 11 = r(K_3 - e, K_7 - K_{1,3})$ and so, by Theorem 1, we have that $r(K_4 - e, K_7 - K_{1,3}) \leqslant 22$, and the result follows. □

The above upper bound improves the previously best known one, given by $r(K_4 - e, K_7 - K_{1,3}) \leqslant 27$.

3.2. Case $m = 5$

The following equality is claimed in [8] without a proof.

Corollary 3. $r(K_5 - e, K_5 - K_{1,3}) = 19$.