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Generalized minor inequalities for the set covering polyhedron related to circulant matrices[☆]

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ABSTRACT

We study the set covering polyhedron related to circulant matrices. In particular, our goal is to characterize the first Chvátal closure of the usual fractional relaxation. We present a family of valid inequalities that generalizes the family of minor inequalities previously reported in the literature and includes new facet-defining inequalities. Furthermore, we propose a polynomial time separation algorithm for a particular subfamily of these inequalities.

1. Introduction

The *weighted set covering problem* can be stated as

$$\text{(SCP)} \quad \min\{c^T x : Ax \geq \mathbf{1}, x \in \{0,1\}^n\}$$

where A is an $m \times n$ matrix with 0, 1 entries, $c \in \mathbb{Z}^n$, and $\mathbf{1} \in \mathbb{R}^m$ is the vector having all entries equal to one. The SCP is a classic problem in combinatorial optimization with important practical applications (crew scheduling, facility location, vehicle routing, to cite a few prominent examples), but hard to solve in general. One established approach to tackle this problem is to study the polyhedral properties of the set of its feasible solutions [5,12,14,15].

The *set covering polyhedron* $Q^*(A)$ is defined as the convex hull of all feasible solutions of SCP. Its *fractional relaxation* $Q(A)$ is the feasible region of the linear programming relaxation of SCP, i.e.,

$$Q(A) := \{x \in [0,1]^n : Ax \geq \mathbf{1}\}.$$

It is known that SCP can be solved in polynomial time if A belongs to the particular class of *circulant matrices* defined in the next section. Hence, it is natural to ask whether an explicit description in terms of linear inequalities can be provided for $Q^*(A)$ in this case, an issue that has been addressed in several recent studies by researchers in the field (see [2,7,8,11] among others). For the related *set packing polytope* of circulant matrices,

$$P^*(A) := \text{conv}(\{Ax \leq \mathbf{1}, x \in \{0,1\}^n\})$$

such a description follows from the results published in [13].

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Bianchi et al. introduced in [7] a family of facet-defining inequalities for $Q^*(A)$ which are associated with certain structures called circulant minors. Moreover, the authors presented in [8] two families of circulant matrices for which $Q^*(A)$ is completely described by this class of *minor inequalities*, together with the full-rank inequality and the inequalities defining $Q(A)$, usually denoted as boolean facets. The existence of a third family of circulant matrices having this property follows from previous results obtained by Bouchakour et al. [9] in the context of the dominating set polytope of some graph classes.

If an inequality $a^Tx \leq b$ is valid for a polytope $P \subset \mathbb{R}^n$ and $a \in \mathbb{Z}^n$, then $a^Tx \leq \lfloor b \rfloor$ is valid for the integer polytope $P_I := \text{conv}(P \cap \mathbb{Z}^n)$. This procedure is called Chvátal–Gomory rounding, and it is known that the system of all linear inequalities which can be obtained in this way defines a new polytope P', the *first Chvátal closure of* P. Moreover, iterating this procedure yields P_I in a finite number of steps. An inequality is said to have *Chvátal rank* of t if it is valid for the tth Chvátal closure of a polytope. All inequalities mentioned above have Chvátal rank less than or equal to one.

With the aim of investigating if the results in [8,9] can be generalized to all circulant matrices, we have tried to characterize the first Chvátal closure of $Q(A)$ for any circulant matrix A. In particular, we addressed the question whether the system consisting of minor inequalities, boolean facets, and the full-rank inequality is sufficient for describing $Q'(A)$. We have obtained a negative answer to this question in the form of a new class of valid inequalities for $Q^*(A)$ which contains minor inequalities as a proper subclass. All inequalities from this class have Chvátal rank equal to one, and besides, some of them define new facets of $Q^*(A)$, as we show by an example.

This paper is organized as follows. In the next section, we introduce some notation and preliminary results required for our work. In Section 3 we describe our approach for computing the first Chvátal closure of $Q(A)$ and define the new class of *generalized minor inequalities*. A separation algorithm for a particular subclass of these is provided in Section 4. Finally, some conclusions and possible directions for future work are discussed in Section 5. A preliminary version of this article appeared without proofs in [16].

2. Notations, definitions and preliminary results

For $n \in \mathbb{N}$, let $[n]$ denote the additive group defined on the set $\{1, \ldots, n\}$, with integer addition modulo n. Throughout this article, if A is a 0, 1 matrix of order $m \times n$, then we consider the columns (resp. rows) of A to be indexed by $[n]$ (resp. by $[m]$). In particular, addition of column (resp. row) indices is always considered to be taken modulo n (resp. modulo m). Two matrices A and A' are *isomorphic*, denoted by $A \approx A'$, if A' can be obtained from A by permutation of rows and columns. Moreover, we say that a row v of A is a *dominating row* if $v \geq u$ for some other row u of A, $u \neq v$.

Given $N \subset [n]$, the *minor of* A *obtained by contraction of* N, denoted by A/N, is the submatrix of A that results after removing all columns with indices in N and all dominating rows. In this work, when we refer to *a minor* of A we always consider a minor obtained by contraction.

Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$, and $C^i := \{i, i+1, \ldots, i+(k-1)\} \subset [n]$ for every $i \in [n]$. With a little abuse of notation we will also use C^i to denote the incidence vector of this set. A *circulant matrix* C_n^k is the square matrix of order n whose ith row vector is C^i. Observe that $C^i = \sum_{j=i}^{i+k-1} e^j$, where e^j is the jth canonical vector in $\mathbb{R}^n$.

A minor of C_n^k is called a *circulant minor* if it is isomorphic to a circulant matrix $C_{n'}^{k'}$. As far as we are aware, circulant minors were introduced for the first time in [11], where the authors used them as a tool for establishing a complete description of ideal and minimally nonideal circulant matrices. More recently, Aguilera [1] completely characterized the subsets N of $[n]$ for which C_n^k/N is a circulant minor.

Example 2.1. Consider the following circulant matrix

$$C_{18}^4 = \begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

and let $N = \{1, 5, 9, 14\}$. The minor C_{18}^4/N obtained from C_{18}^4 by contraction of N is the submatrix of C_{18}^4 that results after removing all columns with indices in N and all dominating rows, which are shown grayed out above. Observe that C_{18}^4/N is isomorphic to C_{14}^3.