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A polyhedral study of the maximum stable set problem with weights on vertex-subsets[☆]

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ABSTRACT

Given a graph $G = (V, E)$, a family of nonempty vertex-subsets $\mathcal{S} \subseteq 2^V$, and a weight $w : \mathcal{S} \to \mathbb{R}_+$, the *maximum stable set problem with weights on vertex-subsets* consists in finding a stable set I of G maximizing the sum of the weights of the sets in $\mathcal{S}$ that intersect I. This problem arises within a natural column generation approach for the vertex coloring problem. In this work we perform an initial polyhedral study of this problem, by introducing a natural integer programming formulation and studying the associated polytope. We address general facts on this polytope including some lifting results, we provide connections with the stable set polytope, and we present three families of facet-inducing inequalities.

1. Introduction

In this work we address a generalization of the maximum weighted stable set problem which arises in the solution of the vertex coloring problem via column generation techniques. Given a graph $G = (V, E)$, a family of nonempty vertex-subsets $\mathcal{S} \subseteq 2^V$, and a weight $w : \mathcal{S} \to \mathbb{R}_+$, we define the *maximum stable set problem with weights on vertex-subsets* (STABws) as the problem of finding a stable set I of G (i.e., a set $I \subseteq V$ such that no two vertices from I are adjacent) that maximizes the sum of the weights of the sets in $\mathcal{S}$ that intersect I. Formally, STABws consists in finding a stable set $I \subseteq V$ maximizing $\sum\{w(S) : S \in \mathcal{S} \text{ and } S \cap I \neq \emptyset\}$. In this context, the vertex subsets in $\mathcal{S}$ are called *structures*. For $v \in V$, define $S(v) = \{S \in \mathcal{S} : v \in S\}$.

STABws naturally arises within the column generation procedure of a straightforward algorithm for the classical vertex coloring problem [2]. In this setting, let $\mathcal{S} \subseteq 2^V$ be the set of maximal stable sets of G. We have a binary variable z_I for every $I \in \mathcal{S}$, and the constraints

$$\sum_{I \in \mathcal{S} : v \in I} z_I \geq 1, \quad v \in V. \tag{1}$$

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In a standard column generation approach, the pricing problem reduces to finding a maximum weighted stable set [3–5]. However, if additional cutting planes are added then the column generation problem no longer corresponds to a classical weighted stable set problem, since the objective function now includes the dual variables corresponding to the added inequalities.

Indeed, let $S_1, \ldots, S_k \subseteq V$ be subsets of vertices and assume that the following cuts have been added to the original formulation:

$$\sum_{I \in \mathcal{S}: I \cap S_i \neq \emptyset} z_I \geq b_i, \quad i = 1, \ldots, k, \tag{2}$$

where $b_1, \ldots, b_k \in \mathbb{R}$ are suitable values making these inequalities valid, i.e., b_i is a lower bound on the minimum number of colors needed to color the subgraph induced by S_i. If λ_v is the dual variable corresponding to Eq. (1), for $v \in V$, and μ_i is the dual variable corresponding to the inequality (2), for $i = 1, \ldots, k$, then the column generation problem consists in finding a stable set $I \subseteq V$ maximizing $\sum\{\lambda_v : v \in I\} + \sum\{\mu_i : S_i \cap I \neq \emptyset, i = 1, \ldots, k\}$. This problem reduces to STABws, where the structures are $\mathcal{S} := \{S_i\}_{i=1}^{k} \cup \{\{v\}\}_{v \in V}$. Note that it is not necessary to consider vertex weights in the definition of STABws, since we can consider a singleton structure for each vertex, thus modeling such weights.

This pricing problem is addressed in [2], where the authors point out that the pricing problem is not exactly a maximum weight stable set problem anymore, give some insights on how the two problems relate to each other, and use these insights in order to improve the bounds of the coloring formulation. Section 6.4 from [2] points out the difficulty to deal with cut generation approaches for the vertex coloring problem (based on the original column generation algorithm in [5]). In [2], the authors attempt to circumvent this difficulty by stating an optimization problem that is "a good approximation of the pricing problem". In practice, such an approximation leads to small improvements in the lower bound, at the cost of a moderate increase in the running times. However, the authors state that the enumeration tree is in most cases significantly smaller. Thus, one can expect that more efficient strategies to solve the pricing problem when cuts are present could help to improve the performance of the whole procedure for solving the vertex coloring problem.

There are two possible approaches for tackling STABws, namely searching for combinatorial algorithms and studying its polyhedral structure with the objective of implementing an algorithm based on integer programming techniques. Combinatorial algorithms for the maximum stable set problem are more effective than integer-programming-based procedures [6,7], but it is not clear how such algorithms can be applied to STABws. This motivates the present work, which aims at providing more details on the structure of the generalized problem and its similarities and main differences with the weighted maximum stable set problem.

In this work we address STABws from an integer programming point of view. We are interested in partial descriptions of the polytope associated with a natural integer programming formulation of this problem. We provide general results on this polytope, including some properties of general facet-inducing valid inequalities, relations between facets of this polytope and facets of the stable set polytope, a straightforward lifting lemma, and a lifting procedure for generating more complex facet-inducing inequalities. We also show how STABws can be reduced to a stable set problem on a larger graph. Although this reduction may not be useful from a practical point of view, it shows additional connections between the polytope studied in this work and the standard stable set polytope. Finally, we present three families of facet-inducing inequalities, two of them being of a quite general nature.

This paper is organized as follows. In Section 2 we present an integer programming formulation of STABws and provide some initial results on the associated polytope $P(G, \mathcal{S})$ (to be defined in Section 2). Section 3 presents a procedure for deriving strong inequalities for $P(G, \mathcal{S})$ from the stable set polytope. Section 4 presents relations between $P(G, \mathcal{S})$ and the stable set polytope, and in Section 5 we explore general families of facets for $P(G, \mathcal{S})$. Finally, Section 6 closes the paper with concluding remarks and open problems.

2. Integer programming formulation

In this section we present an integer programming formulation for STABws. For each vertex $v \in V$, we introduce the binary *vertex variable* x_v such that $x_v = 1$ if and only if the vertex v belongs to the solution. For each structure $S \in \mathcal{S}$, we introduce the binary *structure variable* y_S such that $y_S = 1$ only if the solution intersects S. With these definitions, STABws can be formulated as follows.

$$\max \sum_{S \in \mathcal{S}} w_S y_S$$

$$x_u + x_v \leq 1 \qquad \forall\, uv \in E \tag{3}$$

$$y_S \leq \sum_{v \in S} x_v \quad \forall\, S \in \mathcal{S} \tag{4}$$

$$x_v \in \{0, 1\} \quad \forall\, v \in V \tag{5}$$

$$y_S \in \{0, 1\} \quad \forall\, S \in \mathcal{S}. \tag{6}$$

The objective function asks for the total weight to be maximized. Constraints (3) assert that no two adjacent vertices can be selected (hence x is the characteristic vector of a stable set), whereas constraints (4) assert that $y_S = 0$ if no vertex from S is