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Some results on the structure of kernel-perfect and critical kernel-imperfect digraphs☆

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ABSTRACT

A kernel N of a digraph D is an independent set of vertices of D such that for every $w \in V(D)\setminus N$ there exists an arc from w to N. The digraph D is said to be a kernel-perfect digraph when every induced subdigraph of D has a kernel. Minimal non kernel-perfect digraphs are called critical kernel-imperfect digraphs. The broader sufficient condition for the existence of kernels in digraphs known so far is that states: (1) If D is a digraph such that every odd cycle has two consecutive poles, then D is kernel-perfect.

In this paper is studied the structure of critical kernel-imperfect digraphs which belong to a very large special classes of digraphs and many structural properties are obtained. As a consequence (1) is widely generalized in this class of digraphs, where the condition of the poles is requested only for odd cycles whose edges alternate in a set of arcs. As consequence, some classic results of kernel-perfect and finite critical kernel-imperfect digraphs are generalized for these classes of digraphs.

1. Introduction

The topic of kernels in digraph is a topic widely studied by several authors since it was introduced by Von Neumann and Morgenstern in [24]. It is one of the few generalization of domination in directed graphs. At least two large surveys have been written by Boros and Gurvich [5] and Frankel [11], respectively. The kernels have found many applications, for instance in cooperative n-person games, Nim-type games [3], in logic [2], and recently in computational complexity, artificial intelligence, combinatorics and coding theory. The problem of the existence of a kernel in a given digraph has been studied by several authors, e.g. [1,6–8,16,18,22]. Likewise, the existence of kernels in infinite digraphs has been recently studied by various authors, e.g. [13,21,23]. For extensive information regarding of kernels and their applications, see [19] and [20].

The digraph D is said to be a kernel-perfect digraph when every induced subdigraph of D has a kernel. Minimal non kernel-perfect digraphs are called critical kernel-imperfect digraphs. The existence of a kernel in digraphs, and in particular kernel-perfect orientations of undirected graphs, is strongly related to perfect graphs, and has several applications in combinatorics and game theory [3]. Although the problem of deciding whether a perfect graph has a kernel has been resolved, decision for a plane digraph is NP-complete [10].

In [4], Meyniel conjectured that if every directed cycle of a digraph D has two chords, then D is kernel-perfect. Although this conjecture was shown to be false [12], the searching of a proof motivated the search for sufficient conditions about

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the existence of chords in odd directed cycles to assure that there is a kernel. Some examples of this kind of result are the following:

- **Theorem 1.1** (*Duchet and Meyniel, [9]*). *If every directed cycle has a symmetric arc then D is kernel-perfect.*
- **Theorem 1.2** (*Duchet, [7]*). *If every odd directed cycle has two symmetric arcs then D is kernel-perfect.*
- **Theorem 1.3** (*Duchet, [7]*). *If every odd directed cycle has two short chords and the directed triangle are symmetric then D is kernel-perfect.*
- **Theorem 1.4** (*Galeana-Sánchez and Neumann-Lara, [16]*). *If every odd directed cycle has two pseudodiagonals with consecutives heads or poles on the cycle, then D is kernel-perfect.*

Theorem 1.4 has the so far known widest sufficient condition, regarding the existence of diagonals in odd cycles. In this work, we consider a very broad class of special digraphs and generalized widely Theorem 1.4 for them, proving that the condition of consecutive heads is asked only for certain odd cycles whose edges alternate in a set of arcs. As consequence, some classic results of kernel-perfect and finite critical kernel-imperfect digraphs are generalized for these classes of digraphs.

For the proof of Theorem 1.4, the concept of semikernel was used significantly. The concept of semikernel, given by Neumann-Lara in [21], has been very useful for the searching of sufficient conditions about kernels. Galeana [13] raised a generalization of semikernel, the semikernel modulo F (F is a set of arcs).

Consider the semikernel concept has led to new sufficient conditions for the existence of kernels in digraphs, [13–15].

In this paper, we follow the ideas of the proof of Theorem 1.4, considering the concept of semikernel modulo F, instead of the concept of semikernel. In this manner, we obtain results, which generalize amply this theorem for a wide class of digraphs, digraphs β- and β'-free [14,15]. With the news results are not necessary to ask that all the odd directed cycles have pseudodiagonals with consecutives poles or heads, it will prove that is enough to ask this condition for certain odd directed cycles, which has its arcs turn on the set F.

The critical imperfect digraphs are digraphs without a kernel and every induced proper subdigraph has a kernel. Using a theorem of Neumann-Lara [21], it can prove that a critical kernel-imperfect digraph is a digraph without a kernel and every induced proper subdigraph has a nonempty semikernel.

In [17], important critical kernel-imperfect digraphs' properties are obtained. These properties allowed us to give more sufficient conditions, known so far, for a digraph to be kernel-perfect.

In this paper, we obtain properties of digraphs, which have no kernel and every induced proper subdigraph has a nonempty semikernel modulo F. This kind of digraphs has been studied previously, [13–15]. Every critical kernel-imperfect digraph, where F induced an asymmetrical transitive and the digraph is free of two families (β and β') are digraph of this kind.

Also as consequence we obtain weaker sufficient conditions to assure the existence of kernel and to be a kernel-perfect digraph.

As a consequence, some new structural results concerning finite or infinite critical kernel-imperfect digraphs are obtained, which assure the existence of path and cycles turn on the set F. At the moment we do not know whether or not infinite critical kernel-imperfect digraph exists.

2. Definitions and notations

For general concepts we refer the reader to [3].

Let D be a digraph; $V(D)$ and $A(D)$ will denote the set of vertices and arcs of D, respectively. Let S_1, S_2 be subsets of $V(D)$. The arc u_1u_2 (often we shall write u_1u_2 instead of (u_1, u_2)) will be called an S_1S_2-*arc* whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1. An arc $u_1u_2 \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $u_2u_1 \notin A(D)$ (resp. $u_2u_1 \in A(D)$). The *asymmetrical part* of D which is denoted Asym(D) is the spanning subdigraph of D whose arcs are the asymmetrical arcs of D. A set $I \subseteq V(D)$ is *independent* if $A(D[I]) = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in V(D)\backslash N$ there exists a zN-arc in D. A digraph D is said to be *kernel-perfect* whenever every induced subdigraph of D has a kernel. D is a *critical kernel-imperfect* digraph when D has no kernel but every proper induced subdigraph of D has a kernel.

We write $F^-(S)$ (resp. $F^+(S)$) instead of $\{uv \in A(D) \mid u \notin S, v \in S\}$ (resp. $\{uv \in A(D) \mid u \in S, v \notin S\}$) and F_u^-, F_u^+ for $F^-(\{u\}), F^+(\{u\})$ resp. Let $\Gamma^-(S) = \{v \in V(D)\backslash S \mid \text{ there exists } vu \in A(D), \text{ with } u \in S\}$, $\Gamma^+(S) = \{v \in V(D)\backslash S \mid \text{ there exists } uv \in A(D), \text{ with } u \in S\}$ and Γ_u^-, Γ_u^+ for $\Gamma^-(\{u\}), \Gamma^+(\{u\})$. Let D be a digraph and H a subgraph of D; a *pseudodiagonal* of H is an arc $x_ix_j \in A(D) - A(H)$ where $x_i, x_j \in V(H)$. If $x_i \in U$ and $x_j \in W$, $U, W \subset V(D)$, the arc x_ix_j is an UW-*pseudodiagonal*.

Some previous concepts are used:

Definition 2.1 (*Neumann-Lara, [21]*). Let D be a digraph. A **semikernel** S of D is an independent set of vertices such that for every $z \in V(D)\backslash S$ for which there exists an (S, z)-arc there also exists a (z, S)-arc.