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On basic chordal graphs and some of its subclasses

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ABSTRACT

Basic chordal graphs arose when comparing clique trees of chordal graphs and compatible trees of dually chordal graphs. They were defined as those chordal graphs whose clique trees are exactly the compatible trees of its clique graph.

In this work, we consider some subclasses of basic chordal graphs. One of them is the class of hereditary basic chordal graphs, which will turn out to have many possible characterizations. Those characterizations will show that the class was already studied, but under different names and in different contexts.

We also study the connection between basic chordal graphs and some subclasses of chordal graphs with special clique trees, like *DV* graphs and *RDV* graphs. As a result, it will be possible to define the classes of basic *DV* graphs and basic *RDV* graphs.

Additionally, we study the behavior of the clique operator over all the considered subclasses.

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1. Introduction

1.1. Definitions

For a graph G, we denote the set of its *vertices* by $V(G)$ and the set of its *edges* by $E(G)$. The subgraph *induced* by a subset A of $V(G)$, denoted by $G[A]$, has A as vertex set, and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G. We say that A is a *complete set* of G if $G[A]$ is a complete graph, i.e., all its vertices are pairwise adjacent. A *clique* is a maximal subset of pairwise adjacent vertices, that is, a maximal complete set. The family of cliques of G is denoted by $\mathcal{C}(G)$. For $v \in V(G)$, the family of cliques containing v is denoted by $\mathcal{C}_v$. The reader must be aware of the fact that many papers use the term clique to refer to complete (not necessarily maximal) sets. Thus, a clique in this paper is equivalent to a maximal clique in those other papers.

For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N(v)$ or $N_G(v)$, is the set of all the vertices adjacent to v in G. The *degree* $deg(v)$ of v is the number $|N(v)|$. The *closed neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is the set $N(v) \cup \{v\}$. Vertex v is said to be *simplicial* if $N[v]$ is complete. This is equivalent to $N[v]$ being a clique. Any clique that is the closed neighborhood of some vertex is called *simplicial clique*.

Given two nonadjacent vertices u and v in the same connected component of G, a uv-*separator* is a set S contained in $V(G)$ such that u and v are in different connected components of $G - S$, where $G - S$ denotes the induced subgraph $G[V(G) \setminus S]$. This separator S is *minimal* if no proper subset of S is also a uv-separator. We will just say *minimal vertex separator* to refer to a set S that is a uv-minimal separator for some pair of nonadjacent vertices u and v in G. The family of all minimal vertex separators of G will be denoted by $\mathcal{S}(G)$.

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Let T be a tree. For $v, w \in V(T)$, the notation $T[v, w]$ is used to denote either the path in T from v to w or the vertices of that path, depending on the context. The set of inner vertices of this path is denoted by $T(v, w)$.

Let $\mathcal{F}$ be a family of nonempty sets of vertices of G. If $F \in \mathcal{F}$, then F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F\in\mathcal{F}} F$, then we say that v is a *vertex* of $\mathcal{F}$. The family $\mathcal{F}$ is *Helly* if the intersection of all the members of every subfamily of pairwise intersecting sets is not empty. If $\mathcal{C}(G)$ is a Helly family, then we say that G is a *clique-Helly graph*. We say that $\mathcal{F}$ is *separating* if, for every ordered pair (v, w) of vertices of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The *intersection graph* of $\mathcal{F}$, denoted $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$. The *two-section graph* $S(\mathcal{F})$ of $\mathcal{F}$ is another graph whose vertices are the vertices of $\mathcal{F}$, in such a way that two vertices v and w are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{v, w\} \subseteq F$.

For every vertex v of $\mathcal{F}$, let $D_v = \{F \in \mathcal{F} : v \in F\}$. The *dual family* $D\mathcal{F}$ of $\mathcal{F}$ consists of all the sets D_v. When $\mathcal{F} = \mathcal{C}(G)$, we have that $D_v = \mathcal{C}_v$. An even more general notation will also be used: given a set A of vertices, $\mathcal{C}_A$ is defined to be equal to $\{C \in \mathcal{C}(G) : A \subseteq C\}$.

All graphs considered will be assumed to be connected, unless stated otherwise.

1.2. Chordal graphs, basic chordal graphs and goals

Chordal graphs were originally defined as those graphs for which every cycle of length greater than or equal to four has a *chord*, i.e., an edge connecting two nonconsecutive vertices of the cycle. It was later found that they can be characterized in many other ways. One of them involves the clique tree. A *clique tree* of a graph G is a tree T whose vertex set is $\mathcal{C}(G)$ and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ induces a subtree of T. Alternatively, it can be defined as a tree T whose vertices are the cliques of G and such that, for all $C_1, C_2 \in \mathcal{C}(G)$ and C_3 in $T[C_1, C_2]$, we have $C_1 \cap C_2 \subseteq C_3$. Chordal graphs can be characterized using clique trees as follows,

Theorem 1.1 ([9]). *A graph is chordal if and only if it has a clique tree.*

It is interesting to note that every clique tree of a chordal graph G is a spanning tree of $K(G)$. To prove it, suppose to the contrary that T has an edge CC' that is not an edge of $K(G)$, that is, C and C' are such that $C \cap C' = \emptyset$. Let T_1 and T_2 be the two connected components of $T - CC'$, with $C \in V(T_1)$ and $C' \in V(T_2)$. Since no $v \in V(G)$ is such that $\{C, C'\} \subseteq \mathcal{C}_v$, we can partition $V(G)$ into two sets A and B, where $A = \{v \in V(G) : \mathcal{C}_v \subseteq V(T_1)\}$ and $B = \{v \in V(G) : \mathcal{C}_v \subseteq V(T_2)\}$. Both sets are not empty because C is contained in A and C' is contained in B. Thus, no vertex of A is adjacent to a vertex of B because there is no clique in G containing vertices of both A and B. As a consequence, G would be disconnected, contrary to our initial assumption that we would work with connected graphs only.

Another classical characterization of chordal graphs states that a graph is chordal if and only if every minimal separator of two nonadjacent vertices is a complete set [7]. However, no minimal vertex separator of a chordal graph is a clique. There is an important connection between minimal vertex separators and clique trees that will be reflected in the next three theorems, which are stated here due to their ulterior usefulness.

Given a graph G, two cliques C_1 and C_2 are a *separating pair* if $C_1 \cap C_2$ is a separator of every pair v, w of vertices such that $v \in C_1 \setminus C_2$ and $w \in C_2 \setminus C_1$.

Theorem 1.2 ([13]). *Let G be a chordal graph and $S \in \mathcal{S}(G)$. Then, there exists a separating pair C_1, C_2 such that $S = C_1 \cap C_2$.*

Theorem 1.3 ([13]). *Let C_1 and C_2 be two distinct cliques of a chordal graph G. Then, there exists a clique tree T of G such that $C_1C_2 \in E(T)$ if and only if C_1 and C_2 form a separating pair.*

Finally, it is interesting to note that, when just one clique tree of a graph is known, it is possible to determine what the edges of the other clique trees (if any) can be:

Theorem 1.4 ([13]). *Let G be a chordal graph, T be a clique tree of G and $C_1, C_2 \in \mathcal{C}(G)$, with $C_1 \neq C_2$. Then, there exists a clique tree of G having C_1C_2 as an edge if and only if there exist two cliques C_3 and C_4 that are adjacent in $T[C_1, C_2]$ and with $C_3 \cap C_4 = C_1 \cap C_2$. In that case. $T - C_3C_4 + C_1C_2$ is also a clique tree of G.*

The clique graphs of chordal graphs form an also well known class: *dually chordal graphs*. Dually chordal graphs also have a representative tree structure. A *compatible tree* of a graph G is a spanning tree T of G such that every clique of G induces a subtree in T. The compatible tree can also be defined using the condition that every closed neighborhood of G induces a subtree of T. A graph is dually chordal if and only if it has a compatible tree [2].

In case that we use the definition of compatible tree involving closed neighborhoods, it can be proved that the clique graph of a chordal graph G is dually chordal by showing that any clique tree of G is a compatible tree of $K(G)$.

Proposition 1.5 ([6]). *Let G be a chordal graph. Then, every clique tree of G is compatible with $K(G)$.*

However, it is not necessarily true that every compatible tree of $K(G)$ is a clique tree of G. Consider for example the graph of Fig. 1, which has cliques $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$, $C = \{2, 4, 5\}$ and $D = \{3, 5, 6\}$. Thus $K(G)$ is the complete graph

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