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Near-linear-time algorithm for the geodetic Radon number of grids ☆

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ABSTRACT

The Radon number of a graph is the minimum integer r such that all sets of at least r of its vertices can be partitioned into two subsets whose convex hulls intersect. Determining the Radon number of general graphs in the geodetic convexity is NP-hard. In this paper, we show the problem is polynomial for d-dimensional grids, for all $d \geq 1$. The proposed algorithm runs in near-linear $\mathcal{O}(d\,(\log d)^{1/2})$ time for grids of arbitrary sizes, and in sub-linear $\mathcal{O}(\log d)$ time when all grid dimensions have the same size.

1. Introduction

The concept of convexity in graphs was borrowed from its most well-known geometric counterpart, where a subset S of the Euclidean space is dubbed convex if, for every two points $x, y \in S$, the interval consisting of the straight segment connecting x and y is entirely contained in S. Formally, a *convexity space* (or simply *convexity*, for short) ϕ consists of a pair $(V, \mathcal{C})$, where V is a set – the *ground set* – and $\mathcal{C}$ is a collection of subsets of V – the *convex sets* – such that $\mathcal{C}$ contains both V and the empty set, and $\mathcal{C}$ is closed under arbitrary intersections and nested unions.

The ground set of a geometric convexity is a set of points. Analogously, the ground set of a graph convexity is the set of vertices of some connected graph G. Many different geometric convexities have been studied to date (see, for instance, [3]) and the interval defined by two points is not always the straight segment between them in Euclidean fashion. Likewise, several distinct types of graph convexities have been considered in the literature [8], with applications ranging from statistical physics and distributed computing to marketing and social networks. In the *geodetic* graph convexity – also known as *geodesic* convexity [6] – a set $S \subseteq V(G)$ is convex if, for all $x, y \in S$, every *shortest path* between x and y in G is entirely contained in S.

Given a subset V' of a ground set V, the *convex hull* of V', denoted $[V']$, is the unique minimal convex subset of V containing V'. In the early 1920s, Johann Radon formulated a celebrated theorem stating that every set with at least $d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets whose convex hulls intersect [7]. A natural question concerns what happens when we consider some general ground set V instead of $\mathbb{R}^d$, and the *Radon number* of V is defined as the minimum integer r such that

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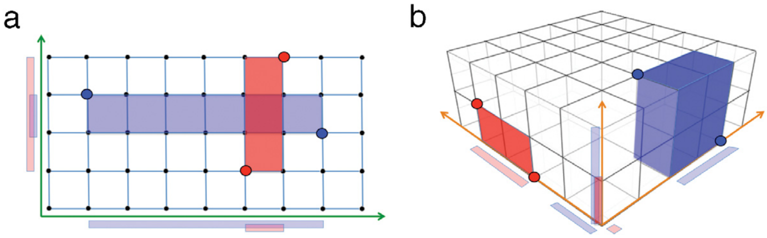

Fig. 1. (a) Radon partition of a set containing four vertices of a 2-dimensional grid: on both dimensions the convex hulls of the projections of the partite sets intersect; (b) *not* a Radon partition of a set containing four vertices of a 3-dimensional grid: there are dimensions in which the projections of one of the partite sets appear all strictly to the left of the projections of the other partite set.

every subset of V with at least r elements can be partitioned in two sets whose convex hulls intersect. The Radon number of graphs has been used to model some problems occurring, for instance, in social networks.

A simple reduction from the maximum clique problem can be used to prove the NP-hardness of finding the Radon number of a graph in the geodetic convexity, hence a natural task is to determine such parameter for particular graph classes. We are interested in the class of d-dimensional grids, i.e., the Cartesian products of d paths of arbitrary sizes. A lot of insight on the problem was gained in [1]. In that paper, the authors derived general bounds and solved the problem for special cases. Computer-assisted results also disclosed (by brute force) the Radon number of all grids up to the ninth dimension. However, a polynomial-time algorithm for determining the Radon number of general grids was still outstanding.

In this paper, we introduce one such algorithm that runs in $\mathcal{O}(d\,(\log d)^{1/2})$ time. Additionally, a variation of our algorithm runs in sub-linear $\mathcal{O}(\log d)$ time for grids $G = P_n^d$, that is, those in which all d dimensions have the same size n.

2. The basics

If R is a subset of vertices of a graph G, then a partition $R = R_1 \cup R_2$ is a *Radon partition* if $[R_1] \cap [R_2] \neq \emptyset$. A set which admits no Radon partitions is called an *anti-Radon set*, also called a *Radon-independent set* by some authors. The Radon number of a graph G is thus the size of the maximum anti-Radon set of G plus one.

A grid $G = \mathrm{Grid}(n_1, \ldots, n_d)$ is the Cartesian product of d paths $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$. The geodetic convexity on d-dimensional grids bears a natural resemblance with the convexity defined on the Euclidean space $\mathbb{R}^d$ by the Manhattan metric $(u, v) \mapsto \|u - v\|_1$. If $R = \{u^1, \ldots, u^r\}$ is a set of vertices of $\mathrm{Grid}(n_1, \ldots, n_d)$ with $u^i = (u_1^i, \ldots, u_d^i)$ for $i \in [1, r] := \{i \in \mathbb{N} : 1 \leq i \leq r\}$, then it is easy to see that the convex hull $[R]$ is the set of integer points in

$$\left[\min_{i\in[1,r]} u_1^i, \max_{i\in[1,r]} u_1^i\right] \times \left[\min_{i\in[1,r]} u_2^i, \max_{i\in[1,r]} u_2^i\right] \times \cdots \times \left[\min_{i\in[1,r]} u_d^i, \max_{i\in[1,r]} u_d^i\right]. \tag{1}$$

In other words, $[R]$ equals the Cartesian product of the one-dimensional convex hulls of the projections of R onto the d dimensions.

Having observed that, one can check whether a partition $R = R_1 \cup R_2$ is a Radon partition by simply inspecting the projections of R onto each dimension. If, for some $j \in [1, d]$, the greatest (smallest) coordinate of the projection of R_1 onto dimension ρ_j is less (greater) than the smallest (greatest) coordinate of the projection of R_2 on ρ_j, then the one-dimensional convex hulls of the projections of R_1 and R_2 onto ρ_j do not intersect, and $[R_1] \cap [R_2] = \emptyset$. In this case, we say the projections of R_1 onto ρ_j appear *all strictly to the left* (*all strictly to the right*) of the projections of R_2. If there is no such j, then the convex hulls of R_1 and R_2 intersect, and R is a Radon partition. Fig. 1 illustrates the idea.

The problem of determining the Radon number of a grid G looks much harder. Indeed, not only is the number of partitions of a given set R exponential in $|R|$, but also the number of subsets R of the ground set $V(G)$ that would have to be checked in the worst case is exponential in $|V(G)|$.

In [2], Eckhoff determined the Radon number of the convexity space defined on $\mathbb{R}^d$ by the Manhattan metric $(u, v) \mapsto \|u - v\|_1$ as

$$r(d) := \min \left\{ r \in \mathbb{N} : \binom{r}{\lfloor \frac{r}{2} \rfloor} > 2d \right\}. \tag{2}$$

In [4], Jamison-Waldner observed that Eckhoff's result could be instantly leveraged to $\mathrm{Grid}(n_1, \ldots, n_d)$ provided $n_j \geq r(d) - 1$, for all $j \in [1, d]$. However, if the grid dimensions are not as large, Eckhoff's result gives an upper bound (which may not be tight). In the next section, we obtain the exact geodetic Radon number of grids. We exploit the following theorem, rewritten from [1], which characterizes grids with anti-Radon sets of size r in a suitable manner.

An *ordered partition* of a set V is a tuple $\psi = (V^1, \ldots, V^n)$, where $V^1 \cup \cdots \cup V^n$ is a partition of V.

Theorem 1 ([1]). *Let $d, n_1, \ldots, n_d, r \in \mathbb{N}$. The graph $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ has an anti-Radon set R with r vertices if and only if the set $[1, r]$ admits d ordered partitions $(V_1^1, \ldots, V_1^{n_1}), (V_2^1, \ldots, V_2^{n_2}), \ldots, (V_d^1, \ldots, V_d^{n_d})$ such that, for every subset S of $[1, r]$ with $1 \leq |S| \leq r/2$, there are indices $j \in [1, d]$ and $\ell \in [1, n_j]$ satisfying either $S = V_j^1 \cup \cdots \cup V_j^\ell$ or $S = V_j^{\ell+1} \cup V_j^{\ell+2} \cup \cdots \cup V_j^{n_j}$.*