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Edge separators for quasi-binary trees

Jorge Luis Ramírez Alfonsín [a,*], Serge Tishchenko [b]

[a] Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2, Place Eugène Bataillon, 34095 Montpellier, France
[b] Faculty of Economics, Lomonosov Moscow State University, Leninskiye gory 1-3, 119991 Moscow, Russian Federation

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ABSTRACT

One wishes to remove $k-1$ edges of a vertex-weighted tree T such that the weights of the k induced connected components are approximately the same. How well can one do it? In this paper, we investigate such k-separators for *quasi-binary* trees. We show that, under certain conditions on the total weight of the tree, a particular k-separator can be constructed such that the smallest (respectively the largest) weighted component is lower (respectively upper) bounded. Examples showing optimality for the lower bound are also given.

1. Introduction

The seminal paper by Lipton and Tarjan [2] has inspired a number of separator-type problems and applications (we refer the reader to [3] for a recent survey on separators and to [4] for edge separators of graphs with bounded genus).

Let us consider the following question.

> One wishes to split a given embedding of a planar connected graph G into blocks formed by weighted faces (weights might be thought of as the areas of the faces) such that the dual of the planar graph induced by each block is connected and the blocks' weights are approximately the same. How well can this be done?

One way to answer the latter is by considering k-separators on a spanning tree T_G of the vertex-weighted dual graph of G. Indeed, one may want to remove $k-1$ edges of T_G such that the weights of the k induced connected components of T_G are approximately the same.

More formally, let $T=(V,E)$ be a graph, and let $\omega : V(T) \longrightarrow \mathbb{R}^+$ be a *weight* function. Let $\omega(T)=\sum_{v\in V(T)}\omega(v)$, and let $2\le k\le |V|-1$ be an integer. A *k-separator* of T is a set $F\subset E(T)$ with $|F|=k-1$ whose deletion induces k connected components, say $C_1(F),\ldots,C_k(F)$. If we let $\omega(C_i(F))=\sum_{v\in V(C_i(F))}\omega(v)$ then $\omega(T)=\sum_{i=1}^{k}\omega(C_i(F))$. Let

$$\beta_k(T) := \max_{F\subseteq E,|F|=k-1}\left\{\min_{1\le i\le k}\omega(C_i(F))\right\}$$

and

$$\alpha_k(T) := \min_{F\subseteq E,|F|=k-1}\left\{\max_{1\le i\le k}\omega(C_i(F))\right\}.$$

* Corresponding author.
E-mail addresses: jramirez@math.univ-montp2.fr (J.L. Ramírez Alfonsín), tichtch@yahoo.fr (S. Tishchenko).

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An optimal k-separator is achieved when $\beta_k(T) = \alpha_k(T) = \frac{1}{k}\omega(T)$.

In this paper, we investigate the existence of k-separators with large (resp. small) values for β_k (resp. for α_k) for the class of *quasi-binary* trees. A tree is called *binary* if the degree of any vertex equals three except for *pendant* vertices (vertices of degree one) and a *root* vertex (a vertex of degree two). A tree is say to be *quasi-binary* if it is a connected subgraph of a binary tree. Notice that good k-separators in quasi-binary trees will lead to good k-block separators for triangulated planar graphs in the above question.

Since, for any quasi-binary tree T, the degree $d(v)$ of any $v \in V(T)$ is 1, 2 or 3, we may define, for each $i = 1, 2, 3$,

$$V_i := \{v \in V(T) | d(v) = i\}$$

and

$$\omega_i := \max\{\omega(v) | v \in V_j \text{ for each } i \le j \le 3\}.$$

We will suppose that $V_2,\ V_3 \neq \emptyset$ and therefore $\omega_2,\ \omega_3 > 0$. Notice that

$$V(T) = V_1 \cup V_2 \cup V_3, \qquad \omega_1 \ge \omega_2 \ge \omega_3 \quad \text{and} \quad \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \ge \omega(T) \tag{1}$$

where $n_i = |V_i|$ for each $i = 1, 2, 3$.

Our main results are the following.

Theorem 1. *Let T be a quasi-binary tree. Let $k \ge 2$ be an integer and $\gamma \in \mathbb{R}$ with $\gamma \ge \omega_3 > 0$. Let $M_\gamma = \max\{\omega_1 + \gamma, 2\omega_2\}$. If*

$$\omega(T) + k\gamma \ge \frac{(k+1)(k-2)M_\gamma}{(k-1)}$$

then

$$\alpha_k(T) \le \frac{2\omega(T) + (k-1)\gamma}{k+1}.$$

Theorem 2. *Let T be a quasi-binary tree. Let $k \ge 2$ be an integer and $\gamma \in \mathbb{R}$ with $\gamma \ge \omega_3 > 0$. Let $M_\gamma = \max\{\omega_1 + \gamma, 2\omega_2\}$. If*

$$\omega(T) + k\gamma \ge \frac{(2k+1)M_\gamma}{(2)}$$

then

$$\beta_k(T) \ge \frac{\omega(T) - (k-1)\gamma}{2k-1}.$$

We note that the bounds for $\alpha_k(T)$ and $\beta_k(T)$ are not necessarily reached by using the same k-separator. The second author has studied k-separators in a more general setting (for planar graphs with weights on vertices, edges, and faces), where a lower bound for β_k is obtained [5]. We noticed that the conditions given in [5] are different from those presented in Theorem 2, whose proof is distinct (on the same line as that of Theorem 1). The value α_k is not treated in [5] at all. Two-separators for binary trees were also studied in [1, Corollary 2.2] where it is proved that a binary tree T with at least $\lambda + 1$ vertices admits an edge separating a forest F in T satisfying $\lambda \le |V(F)| < 2\lambda$ for any real number $\lambda > \frac{1}{2}$. In the same spirit, Theorem 1 (resp. Theorem 2) implies, by taking $\omega(v) = 1$ for all vertices v and $\gamma = \lambda - 2$ (resp. $\gamma = \lambda$), the existence of a 2-separator of a binary tree with $\lambda \ge 1$ vertices such that one of the two connected components has at most λ (resp. at least $\frac{2}{3}\lambda$) vertices.

In the following section, we present some preliminary results needed for the rest of the paper. Main results are proved in Section 3. Finally, a family of quasi-binary trees showing optimality of Theorem 2 is constructed in the last section.

2. Preliminary results

Let T be a quasi-binary tree. We may suppose $\omega(T) > 0$ and that $n = |V(T)| > 1$ henceforth. For $i \in \{1, 2, 3\}$, let $n_i = |V_i|$. We observe that

$$n_1 + 2n_2 + 3n_3 = 2|E(T)| = 2(n-1) = 2n_1 + 2n_2 + 2n_3 - 2 \quad \text{and thus } n_1 = n_3 + 2. \tag{2}$$