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# A note on the middle levels problem[☆]

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ABSTRACT

The middle levels problem consists in determining a hamiltonian cycle in the bipartite Kneser graph $B(2k+1, k)$, also known as the middle levels graph and denoted by $B_k$. Previously, it was proved that a particular hamiltonian path in a reduced graph of $B_k$ implies a hamiltonian cycle in $B_k$ and a hamiltonian path in the Kneser graph $K(2k+1, k)$. We show that the existence of such a particular hamiltonian path in a reduced graph of $K(2k+3, k)$ implies a hamiltonian path in $K(2k+3, k)$ for $k \equiv 1$ or $2 \pmod 3$. Moreover, we utilize properties from the middle levels graphs to improve a known algorithm speeding up the search for such a particular hamiltonian path in the reduced graph of $B_k$.

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## 1. Introduction

Let $\mathbb{Z}_n$ be the set $\{1, \ldots, n\}$ and let $\binom{\mathbb{Z}_n}{k}$ be the family of all $k$-subsets of $\mathbb{Z}_n$. The *Kneser graph* $K(n, k)$ has $\binom{\mathbb{Z}_n}{k}$ as its vertex set, and two $k$-subsets are adjacent if they are disjoint (Fig. 1(a) and (b)). The Kneser graph $K(2k+1, k)$ is called the *odd graph* and it is denoted by $O_k$ (Fig. 1(a)). Notice that $O_2$ is the Petersen graph.

The *bipartite Kneser graph* $B(n, k)$ has $\binom{\mathbb{Z}_n}{k} \cup \binom{\mathbb{Z}_n}{n-k}$ as its vertex set and an undirected edge $\{u, v\}$ represents that $u$ either contains or is contained into $v$ (Fig. 1(c)). The graph $B(2k+1, k)$ is isomorphic to the subgraph of the $(2k+1)$-cube graph induced by the vertices having exactly $k$ or $(k+1)$ ones. Hence, $B(2k+1, k)$ is also called the *middle levels graph*, and it is denoted by $B_k$.

A long standing conjecture due to Lovász [12] claims that every connected vertex-transitive graph has a hamiltonian path. For $n \geq 2k+1$, the graphs $K(n, k)$ and $B(n, k)$ form well-studied families of connected vertex-transitive graphs, both $\binom{n-k}{k}$-regular. A conjecture due to Biggs [1] claims that the odd graphs are hamiltonian for $k \geq 3$ and a related conjecture by Havel [8] claims that the middle-layers graph is hamiltonian. After more than 30 years, a proof of Havel's conjecture has been announced [14]. Note that, if the Kneser graph $K(n, k)$ has a hamiltonian cycle $(A_1, A_2, A_3, \ldots, A_t)$ where $t = |V(K(n, k))|$ and $t$ is odd, then $(\overline{A_1}, A_2, \overline{A_3}, \ldots, A_{t-1}, \overline{A_t}, A_1, \overline{A_2}, A_3, \ldots, \overline{A_{t-1}}, A_t)$ is a hamiltonian cycle of the bipartite Kneser graph $B(n, k)$. It seems that Havel's conjecture might be slightly easier to prove than Biggs' conjecture, as unlike $O_k$ the graph $B_k$ is bipartite. At the moment, Biggs' conjecture remains unsolved.

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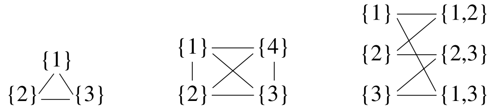


(a) $K(3,1) = O_1$. (b) $K(4,1)$. (c) $B(3,1) = B_1$.

**Fig. 1.** For $k = 1$, the graphs: (a) $O_k$; (b) $K(2k+2, k)$, and; (c) $B_k$.

Apart from the Petersen graph, $K(n, k)$ is hamiltonian if $n \geq (3k + 1 + \sqrt{5k^2 - 2k + 1})/2$, as reported in [5], or if $n \leq 27$ (see [17] and references therein). Recently, Johnson [10] showed how to construct hamiltonian cycles in $K(2k + i, k)$ for $k \leq 2l + 1$ using hamiltonian cycles from $K(2k + \frac{i}{2}, k)$ for $k \leq l$. This implies that, since $O_k$ is hamiltonian for $3 \leq k \leq 13$, then $K(2k+2, k)$ is hamiltonian for $k \leq 27$ and, consequently, $K(2k+4, k)$ is hamiltonian for $k \leq 55$, and so on. In particular, for $n = 2k + 1$, it is known that $O_k$ has a hamiltonian path or a hamiltonian cycle for $k \leq 19$ [2,19]. The strategy from [10] cannot be applied to $K(2k + 3, k)$.

According to [16], in an unpublished work, Moews and Reid have determined that $B_k$ is hamiltonian when $k \leq 11$. In their work, they have used a reduced graph of the middle levels graph $B_k$, smaller than $B_k$ by a factor of $2n$. Part of the reduction technique had been previously used by Dejter [6]. Later, the reduced graph of the odd graph $O_k$ was introduced [2]. It was proved that a particular hamiltonian path – we shall call it a *useful path* – in the reduced graph of $B_k$ and $O_k$ provides a hamiltonian cycle and path in $B_k$ and $O_k$, respectively [2,16]. In the extended abstract [3], we have established that a useful path in the reduced graph of $K(2k + 2, k)$ implies a hamiltonian cycle in both graphs $B(2k + 2, k)$ and $K(2k + 2, k)$. We observe that the announced proof [14] for Havel's conjecture does not use a useful path in a reduced graph, but instead proposes to modify a 2-factor locally by flippable pairs to get a single cycle.

At the moment, all known useful paths have been determined by heuristic algorithms. First, a heuristic [16] based on Pósa's path reversal strategy [15] was proposed to search for useful paths in the reduced graph of $B_k$, and had its running time further improved by Shields et al. [18]. Shimada and Amano [19] proposed a partition of the vertices of the reduced graph of $B_k$ such that the existence of a path in each set of vertices implies a useful path. They have used the algorithm in [16] to determine useful paths for $k = 18, 19$.

In Section 2, we show that a useful path in the reduced graph of $K(2k + 3, k)$ implies a hamiltonian path in $K(2k + 3, k)$ for $k \equiv 1$ or $2 \pmod 3$. In Section 4, we propose an improvement on the algorithm in [16] speeding up the search for useful paths in the reduced graphs of the middle levels graph. Since a useful path in the reduced graph of the middle levels graph gives not only a hamiltonian cycle in $B_k$ but also a hamiltonian path in the odd graph $O_k$, our results also contribute to Biggs' conjecture. This improvement is provided by using modular matchings [7], a 1-factorization of $B_k$ that has been used in hamiltonicity results in Kneser and Bipartite Kneser graphs [4,9,13]. Experimental results are presented in Section 5. We conclude with some remarks and open problems in Section 6.

## 2. Useful paths in the reduced graph of $K(2k + 3, k)$

Consider $\mathbb{Z}_n$ with arithmetical operations modulo $n$, and the vertices of $K(n = 2k + i, k)$ as $k$-subsets of $\mathbb{Z}_n$.

Let $r_1 = \{1, 2, \ldots, k\}$ and $r_2 = \{2, 4, 6, \ldots, 2k = n - i\}$ two $k$-subsets of $\mathbb{Z}_n$. Given a set $A \subseteq \mathbb{Z}_n$ and an integer $\delta \in \mathbb{Z}$, define $A + \delta$ as the set $\{a + \delta : a \in A\}$ and $\overline{A}$ as the set $\mathbb{Z}_n \setminus A$. Define the equivalence relation $\sim$ as follows: given $A, B \subset \mathbb{Z}_n$, $B \sim A$ if either: (i) $B = A + \delta$ or (ii) $\overline{B} = A + \delta$. Refer to the equivalence class of $A$ under $\sim$ as $\sigma(A)$. We call $A + 1$ a *shift* of $A$, denoted by $sh(A)$. Let $sh^0(A) = sh^n(A) = A$, and $sh^{t+1}(A) = sh^t(sh(A))$ for $t \geq 0$. If $C = (v_1, \ldots, v_j)$ is a path or a cycle, then let $sh^t(C) = (sh^t(v_1), \ldots, sh^t(v_j))$.

Given a graph $G$, the *reduced graph* $R(G)$ is the graph obtained from $G$ by identifying vertices that are equivalent according to $\sim$. More precisely, the vertices of $R(G)$ are the equivalence classes $\sigma(v)$, for each $v \in V(G)$, and if $\{u, v\} \in E(G)$ then $\{\sigma(u), \sigma(v)\} \in E(R(G))$. Notice that if $\{u, v\} \in E(G)$ satisfies $u \sim v$, then the vertex $\sigma(u) \in V(R(G))$ has a loop. Fig. 2(a) shows the graph $R(K(8, 3))$, which is the reduced graph of $K(8, 3)$ (Fig. 2(b)).

Shields and Savage [16] showed that the existence of a hamiltonian path that starts with vertex $\sigma(r_1)$ and ends with vertex $\sigma(r_2)$ in the graph $R(B_k)$ implies that $B_k$ is hamiltonian. We refer to a hamiltonian path starting with $\sigma(r_1)$ and ending with $\sigma(r_2)$ as a *useful path*.

The property of a useful path in the reduced graph implying a hamiltonian cycle or path in connected Kneser and bipartite Kneser graphs seems to be true whenever $k$ and $n$ are relatively prime. It has already been proved that, for $n = 2k + 1$ or $n = 2k + 2$, a useful path in $R(B(n, k))$ implies a hamiltonian cycle or a hamiltonian path in $B(n, k)$ and $K(n, k)$ [3,2,16]. In this paper, we extend this result for $K(2k+3, k)$. Lemmas 1 to 4 are generalizations from [3,2,16] for any $n \geq 2k+1$, needed to achieve our results.

**Lemma 1.** *For $n \geq 2k + 1$, $R(K(n, k)) = R(B(n, k))$.*

**Proof.** The vertices of $K(n, k)$ are the $k$-subsets of $\mathbb{Z}_n$, and the vertices of $B(n, k)$ are the $k$-subsets and $(n - k)$-subsets of $\mathbb{Z}_n$. Since the complement of a $k$-subset is an $(n - k)$-subset, we have $V(R(K(n, k))) = V(R(B(n, k)))$. There is an edge $\{A, B\} \in E(K(n, k))$ if $A \cap B = \emptyset$, and there is an edge $\{A, \overline{B}\} \in E(B(n, k))$ if $A \subset \overline{B}$. Since $n \geq 2k + 1$ and, therefore, $|A| \neq |\overline{B}|$, both statements are equivalent and $E(R(K(n, k))) = E(R(B(n, k)))$. $\square$

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