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A note on adjacent vertex distinguishing colorings of graphs

M. Axenovich [a,*], J. Harant [b], J. Przybyło [c], R. Soták [d], M. Voigt [e], J. Weidelich [a]

[a] Karlsruher Institut für Technologie, Germany
[b] Technical University of Ilmenau, Germany
[c] AGH University of Science and Technology, Krakow, Poland
[d] Pavol Jozef Šafárik University in Košce, Slovakia
[e] Dresden University of Applied Sciences, Germany

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ABSTRACT

For an assignment of numbers to the vertices of a graph, let $S[u]$ be the sum of the labels of all the vertices in the closed neighborhood of u, for a vertex u. Such an assignment is called *closed distinguishing* if $S[u] \neq S[v]$ for any two adjacent vertices u and v unless the closed neighborhoods of u and v coincide. In this note we investigate $\mathrm{dis}[G]$, the smallest integer k such that there is a closed distinguishing labeling of G using labels from $\{1, \ldots, k\}$. We prove that $\mathrm{dis}[G] \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of G. This result is sharp. We also consider a list-version of the function $\mathrm{dis}[G]$ and give a number of related results.

1. Introduction

One of the important tasks in network studies is to be able to identify and distinguish their elements, e.g. vertices, using local substructures and small labels. Among the multiple ways to achieve this, one of the most natural ones is to distinguish vertices by the sums of labels in their neighborhoods.

Formally, for a graph G, the open neighborhood of a vertex v is denoted $N(v)$ and the closed neighborhood is denoted $N[v]$. We say that an assignment of numbers (labels) to the vertices of G is *closed distinguishing* if the sum of labels of the vertices in $N[v]$ differs from the sum of labels of the vertices in $N[u]$ for any adjacent vertices u and v unless $N[u] = N[v]$. Let $\mathrm{dis}[G]$ be the smallest integer k such that there is a closed distinguishing assignment for G using integers from the set $\{1, \ldots, k\}$. Define also $\mathrm{dis}(G)$ using $N(u)$ instead of $N[u]$ and call the corresponding coloring *open distinguishing*. Note that both values $\mathrm{dis}[G]$ and $\mathrm{dis}(G)$ exist because an assignment of n distinct powers of 2 to the n-vertex graph gives distinct sums for any two subsets of vertices. A better upper bound on $\mathrm{dis}(G)$ might be derived from unpublished Theorem 1 of Norin, see [7] for description. This and other results on $\mathrm{dis}(G)$ in planar graphs can be found in [7,11], where in particular it is conjectured that $\mathrm{dis}(G)$ is at most the chromatic number $\chi(G)$ of G. Recall that the chromatic number is the smallest number assigned to the vertices of a graph such that no two adjacent vertices have the same color, a coloring number of a graph G is the largest minimum degree of a subgraph of G plus 1, see [12].

Theorem 1 (*S. Norin, [23], see [7]*). *Let G be a graph with chromatic number r and coloring number k. Let $n_1, \ldots, n_r$ be pairwise coprime integers with $n_i \geq k$ for $i = 1, \ldots, r$. Then* $\mathrm{dis}(G) \leq n_1 n_2 \ldots n_r$.

* Corresponding author.
E-mail addresses: maria.aksenovich@kit.edu (M. Axenovich), jochen.harant@tu-ilmenau.de (J. Harant), przybylo@wms.mat.agh.edu.pl (J. Przybyło), roman.sotak@upjs.sk (R. Soták), mvoigt@informatik.htw-dresden.de (M. Voigt), jennyweidelich@web.de (J. Weidelich).

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While the asymptotic behavior of dis(G) remains a widely open problem, we focus on the dis$[G]$ function in this note. It is worth mentioning that vertex-distinguishing labelings have already been thoroughly investigated in the case of planar graphs. By taking $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, and $n_4 = 11$ in Theorem 1, it follows that dis$(G) \le 5544$ for a planar graph G. In [7], this bound is improved to 468. Moreover, it is shown there that dis$(G) \le 36$ for a 3-colorable planar graph, that dis$(G) \le 4$ for a planar graph of girth at least 13, and that dis$(G) \le 2$ if G is a tree. The large body of research in this area uses not only the labels on vertices but also on the edges and the faces of a graph. One of the fascinating conjectures in the area is so-called 1–2–3 Conjecture by Karoński, Łuczak, and Thomason posed in [22] as a question. It claims that it is sufficient to use integers 1, 2 and 3 in order to label the *edges* of any non-trivial (of order ≥ 3) connected graph so that adjacent vertices meet distinct sums of their incident labels. This conjecture is true e.g. for 3-colorable graphs, see [22], and it is known that labels chosen from the set $\{1, 2, \ldots, 5\}$ are sufficient, as shown by Kalkowski, Karoński, and Pfender in [21]; see also [1,27], and [2] for earlier results. A number of variations of distinguishing labelings of graphs have also been considered, see for example [3–5,8–10,14–16,18,20,24–26,28]. See also a survey on graph labelings by Gallian [17].

In this note we initiate the study of dis$[G]$, provide the sharp upper bound on this parameter in terms of the maximum degree of a graph and several related results. Our main theorem is stated in a general setting of list-colorings. Assume that every vertex is endowed with a list $L(v) \subseteq \mathbb{R}$ of available labels. Let $\mathrm{dis}_\ell[G]$ ($\mathrm{dis}_\ell(G)$) be the smallest k such that for every list assignment with $|L(v)| \ge k$ for all $v \in V(G)$ there is a closed (open) distinguishing assignment giving every vertex v a label from its list and $\chi_\ell(G)$ is the smallest integer k such that for every list assignment L with $|L(v)| \ge k$ there is a proper vertex coloring c with $c(v) \in L(v)$ for all $v \in V$. Trivially, for complete graph K_n we have $\mathrm{dis}_\ell[K_n] = \mathrm{dis}[K_n] = 1$ and $\mathrm{dis}_\ell(K_n) = \mathrm{dis}(K_n) = \chi_\ell(K_n) = n$.

Below we formulate our results. We prove them in Section 2.

Theorem 2. *Let $\Delta \ge 2$ be an integer and G be a graph with $\Delta(G) = \Delta$. Then*

- $\mathrm{dis}[G] \le \mathrm{dis}_\ell[G] \le \Delta^2 - \Delta + 1$.
- *For every Δ, there is a graph G such that* $\mathrm{dis}[G] \ge (1 - o(1))\Delta^2$.
- *There are infinitely many values of Δ for which G might be chosen so that* $\mathrm{dis}[G] = \Delta^2 - \Delta + 1$.

Theorem 3. *Let $\Delta \ge 2$, $k \ge 2$ be integers and G be a graph with the coloring number k and $\Delta(G) = \Delta$. Then* $\mathrm{dis}_\ell(G) \le (k-1)\cdot\Delta + 1 \le \Delta^2 + 1$.

The coloring number of a planar graph is at most 6. Hence, we have immediately the following Corollary.

Corollary 4. *If G is a planar graph, then* $\mathrm{dis}_\ell(G) \le 5 \cdot \Delta + 1$.

If $\Delta \le 93$ then this result gives a bound that is better than the best bound 468 known so far.

One of the challenging problems in the area is to determine how 'dis' function depends on the chromatic number of a graph. Here, we show that the dis$[G]$ function can be arbitrarily high even for bipartite graphs. In addition, we give some upper bounds depending on the number of edges and degrees. Note that it is not known whether there is a function f such that dis$(G) \le f(\chi(G))$.

Note that it is essential here, that the labels are strictly positive integers. Indeed, if we could use 0 as a label, the situation becomes easy for most bipartite graphs. Assigning 0 to all the vertices of one partite set and a label 1 to all other vertices, gives an open and a closed distinguishing coloring of any bipartite graph in which the vertices in one partite set have degrees at least 2.

Theorem 5.
- *Let G be a bipartite graph with partite sets A and B which is not a star. Let, for $X \in \{A, B\}$, $\Delta_X = \max_{x\in X} d(x)$ and $\delta_{X,2} = \min_{x\in X, d(x)\ge 2} d(x)$, where $d(x)$ denotes the degree of x in G. Then*

$$\mathrm{dis}[G] \le \min\left\{c\sqrt{|E(G)|}, \left\lfloor \frac{\Delta_A - 1}{\delta_{B,2} - 1} \right\rfloor + 1, \left\lfloor \frac{\Delta_B - 1}{\delta_{A,2} - 1} \right\rfloor + 1\right\},$$

  where c is some constant.
- *For every cycle C_n of length $n \ge 4$,* $\mathrm{dis}_\ell[C_n] = \mathrm{dis}[C_n] = \chi(C_n)$.
- *For any positive integer k, there is a bipartite graph G such that* $\mathrm{dis}[G] > k$.

In case of trees we provide results for list, modulo p, and ordinary sum-distinguishing numbers.

Theorem 6. *Let $T \ne K_2$ be a tree.*

- *We have* $\mathrm{dis}_\ell[T] \le 3$ *and* $\mathrm{dis}[T] \le 2$.
- *If $L(v)$, $v \in V(T)$, are lists of 2 positive numbers each, then T has a closed distinguishing labeling from these lists.*
- *Let $p \ge 4$ be an integer. There exists a labeling $w : V(T) \to \{1, 2, 3\}$ such that $\sum_{u\in N[x]} w(u) \not\equiv \sum_{v\in N[y]} w(v) \pmod p$ for all edges $xy \in E(T)$.*

We say that a forest is *strong* if it does not have a K_2 component.

Corollary 7. *Let G be a graph whose edges are covered by k induced strong forests. Let $p_1, \ldots, p_k$ be pairwise relatively prime integers, each at least 4. Then* $\mathrm{dis}[G] \le p_1p_2\ldots p_k$.