Discrete Applied Mathematics 205 (2016) 8–15

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the first-order edge tenacity of a graph

Bahareh Bafandeh [a], Dara Moazzami [b,*], Amin Ghodousian [b]

[a] University of Tehran, Department of Algorithms and Computation, Iran

[b] University of Tehran, College of Engineering, Department of Engineering Science, Iran

ARTICLE INFO

Article history:
Received 29 March 2013
Received in revised form 6 September 2014
Accepted 20 October 2015
Available online 20 February 2016

Keywords:
Edge-tenacity
Planar graph
Balancity

ABSTRACT

The first-order edge-tenacity $T_1(G)$ of a graph G is defined as

$$T_1(G) = min\left\{\frac{|X| + \tau(G-X)}{\omega(G-X)-1}\right\}$$

where the minimum is taken over every edge-cutset X that separates G into $\omega(G-X)$ components, and by $\tau(G-X)$ we denote the order (the number of edges) of a largest component of $G-X$.

The objective of this paper is to study this concept of edge-tenacity and determining this quantity for some special classes of graphs. We calculate the first-order edge-tenacity of a complete n-partite graph. We shall obtain the first-order edge-tenacity of maximal planar graphs, maximal outerplanar graphs, and k-trees. Let G be a graph of order p and size q, we shall call the least integer r, $1 \le r \le p-1$, with $T_r(G) = \frac{q}{p-r}$ the balancity of G and denote it by $b(G)$. Note that the balancity exists since $T_r(G) = \frac{q}{p-r}$ if $r = p-1$. In general, it is difficult to determine the balancity of a graph. In this paper, we shall first determine the balancity of a special class of graphs and use this to find an upper bound for the balancity of an arbitrary graph.

1. Introduction

Throughout this paper, our terminology will be standard except as indicated. We use $V(G)$ and $\omega(G)$ to denote the vertex set and number of components in a graph G, respectively. The concept of tenacity of a graph G was introduced in [6,7], as a useful measure of the "vulnerability" of G. In [7] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs. In [14] we showed a complete proof for case three of the Harary Graphs. In [16], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [1,4,5,11–13,15,16,18,19,17,20,21,14,8,24,25,32,30,31,29,28,33], the authors studied more about this new invariant. The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+m(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $m(G-S)$ to be the number of the vertices in a largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $|S| + m(G-S) \ge T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex

* Correspondence to: School of Computer Sciences, Institute for Research in Fundamental Sciences (IPM), P.O.Box:19395-5746, Tehran, Iran.
E-mail address: dmoazzami@ut.ac.ir (D. Moazzami).

http://dx.doi.org/10.1016/j.dam.2015.10.027

cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+m(G-S)}{\omega(G-S)}$.

The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A \subseteq E(G)} \left\{ \frac{|A| + m(G-A)}{\omega(G-A)} \right\}$$

where $m(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. Cozzens et al. in [6], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.

After the pioneering work of Cozzens, Moazzami, and Stueckle in [6,7], several groups of researchers have investigated tenacity, and its related problems. In [24] and [25] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subseteq E(G)$ and the number of vertices of a largest component, $m(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter did not seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [20]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e(G) = \min_{A \subseteq E(G)} \left\{ \frac{|A| + \tau(G-A)}{\omega(G-A)} \right\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question "How difficult is it to recognize T-tenacious graphs?" has remained an interesting open problem for some time. The question was first raised by Moazzami in [15]. Our purpose in [8] was to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we showed that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET.

For an integer k, $1 \leq k \leq |V(G)| - 1$, we define the *k-order* edge-tenacity of a graph G as

$$T_k(G) = min \left\{ \frac{|X| + \tau(G-X)}{\omega(G-X) - k} | X \subseteq E(G) \text{ and } \omega(G-X) > k \right\}$$

where the minimum is taken over all edge-cutset X of G with $\omega(G-X) > k$.

In [22] and [27], respectively, Nash-Williams and Tutte proved the following theorem.

Theorem A. *A connected graph G has s edge-disjoint spanning trees if and only if*

$$|X| \geq s(\omega(G-X) - 1) \quad \textit{for each } X \subseteq E(G).$$

Thus, as an immediate consequence we have:

Theorem 1. *If a connected graph G has s edge-disjoint spanning trees then*

$$|X| + \tau(G-X) \geq s(\omega(G-X) - 1) \quad \textit{for each } X \subseteq E(G).$$

Motivated by this result, we can introduce the following corollary.

Corollary 1. *If a graph G has s edge-disjoint spanning trees then $T_1(G) \geq s$.*

Corollary 2. *Let G be a graph of order p and size q and let k be an integer with $1 \leq k \leq p-1$, then $T_k(G) \leq \frac{q}{p-k}$.*

Conjecture. *The first-order edge-tenacity of a graph is* NP*-complete.*

It is not clear whether the first-order edge-tenacity of a graph can be computed in polynomial time. However, the maximum number of edge-disjoint spanning trees in a graph can be computed in polynomial time by matroid partitioning algorithms ([9] see also [26]), and so by Corollary 1 the first-order edge-tenacity of a graph can be very closely approximated.

The objective of this paper is to examine and study various classes of graphs for which their first-order edge-tenacity can be readily determined.

We first state the following result:

Theorem 2. *Let G be a graph. Then $\frac{|E(G)|}{(|V(G)|-1)} \leq \frac{|X|+\tau(G-X)}{\omega(G-X)-1}$ for every edge-cutset X of G if and only if $\frac{|E(H)|}{(|V(H)-1|)} \leq \frac{|E(G)|}{(|V(G)|-1)} + \frac{\tau(G-X)}{|V(G)|-\omega(G-X)}$ for every subgraph H of G.*