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Global cycle properties of locally isometric graphs

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A B S T R A C T

Let $\mathcal{P}$ be a graph property. A graph G is said to be *locally* $\mathcal{P}$ if the subgraph induced by the open neighbourhood of every vertex in G has property $\mathcal{P}$. Ryjáček's well-known conjecture that every connected, locally connected graph is weakly pancyclic motivated us to consider the global cycle structure of locally $\mathcal{P}$ graphs, where $\mathcal{P}$ is the property of having diameter at most k for some fixed $k \geq 1$. For $k = 2$ these graphs are called *locally isometric graphs*. For $\Delta \leq 5$, we obtain a complete structural characterization of locally isometric graphs that are not fully cycle extendable. For $\Delta = 6$, it is shown that locally isometric graphs that are not fully cycle extendable contain a pair of true twins of degree 6. Infinite classes of locally isometric graphs with $\Delta = 6$ that are not fully cycle extendable are described and observations are made that suggest that a complete characterization of these graphs is unlikely. It is shown that Ryjáček's conjecture holds for all locally isometric graphs with $\Delta \leq 6$. The Hamilton cycle problem for locally isometric graphs with maximum degree at most 8 is shown to be NP-complete.

1. Introduction

Recent advances in graph theory have been influenced significantly by the rapid growth of the internet and corresponding large communication networks. Of particular interest are social networks such as Facebook where it is not uncommon for the friends, i.e., neighbours, of any given person to be themselves 'closely' connected within this neighbourhood. In this paper we study global cycle structures of graphs for which the neighbourhoods of all vertices induce graphs in which every two nodes are not 'too far apart'. To make these notions more precise we begin by introducing some terminology and background on closely related literature.

Let G be a graph. The order of G is denoted by $n(G)$ and the minimum and maximum degree of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. If G is clear from context we use n, δ and Δ, to denote these respective quantities. We say that G is *Hamiltonian* if G has a cycle of length n. If, in addition, G has a cycle of every length from 3 to n, then G is *pancyclic*. An even stronger notion than pancyclicity is that of 'full cycle extendability', introduced by Hendry [12]. A cycle C in a graph G is *extendable* if there exists a cycle C' in G that contains all the vertices of C and one additional vertex. A graph G is *cycle extendable* if every nonhamiltonian cycle of G is extendable. If, in addition, every vertex of G lies on a 3-cycle, then G is *fully cycle extendable*.

A graph G that is not necessarily Hamiltonian but has cycles of every possible length from the shortest cycle length $g(G)$ (called the *girth* of G) to the longest cycle length $c(G)$ (called the *circumference* of G) is said to be *weakly pancyclic*.

By a *local property* of a graph we mean a property that is shared by the subgraphs induced by all the open neighbourhoods of the vertices. We use $N(v)$ to denote the open neighbourhood of a vertex $v \in V(G)$. If $X \subseteq V(G)$, the subgraph induced by

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X is denoted by $\langle X \rangle$. For a given graph property $\mathcal{P}$, we call a graph G *locally* $\mathcal{P}$ if $\langle N(v) \rangle$ has property $\mathcal{P}$ for every $v \in V(G)$. Skupień [20] defined a graph G to be *locally Hamiltonian* if $\langle N(v) \rangle$ is Hamiltonian for every $v \in V(G)$. Locally Hamiltonian graphs were subsequently studied, for example, in [16,17,21]. A graph is *traceable* if it has a Hamiltonian path. Pareek and Skupień [17] considered locally traceable graphs and Chartrand and Pippert [6] introduced locally connected graphs. The latter have since been studied extensively—see for example [5–7,10,12,11,15].

A classic example of a local property that guarantees Hamiltonicity is Dirac's minimum degree condition '$\delta(G) \geq n(G)/2$' (see [8]), which may be written as '$|N(v)| \geq n(G)/2$ for every vertex v in G'. Bondy [2] showed that Dirac's minimum degree condition actually guarantees more than just the existence of a Hamilton cycle by showing that these graphs are either pancyclic or isomorphic to the complete balanced bipartite graph $K_{n/2,n/2}$.

Another local property that is often studied in connection with Hamiltonicity is the property of being claw-free, i.e., not having the claw $K_{1,3}$ as induced subgraph. Note that a graph G is claw-free if and only if $\alpha(\langle N(v) \rangle) \leq 2$ for every $v \in V(G)$ (where α denotes the vertex independence number).

It is well known that the *Hamilton Cycle Problem* (i.e., the problem of deciding whether a graph has a Hamiltonian cycle) is NP-complete, even for claw-free graphs. However, Oberly and Sumner [15] showed that, connected, locally connected claw-free graphs are Hamiltonian. Clark [7] strengthened this result by showing that under the same hypothesis these graphs are in fact pancyclic and Hendry [12] observed that Clark had in fact shown that these graphs are fully cycle extendable. These results support Bondy's well-known 'meta-conjecture' (see [3]) that almost any condition that guarantees that a graph has a Hamilton cycle usually guarantees much more about the cycle structure of the graph. If the claw-free condition is dropped, Hamiltonicity is no longer guaranteed. In fact, Pareek and Skupień [17] observed that there exist infinitely many connected, locally Hamiltonian graphs that are not Hamiltonian. However, Clark's result led Ryjáček to suspect that every locally connected graph has a rich cycle structure, even if it is not Hamiltonian. He proposed the following conjecture (see [23]).

Conjecture 1 (*Ryjáček*)**.** *Every locally connected graph is weakly pancyclic.*

Ryjáček's conjecture seems to be very difficult to settle. The conjecture for $\Delta \leq 5$ was settled in [22]. However, even for graphs with $\Delta = 6$ this problem is largely unsolved. Weaker forms of this conjecture are considered in [22] where it was shown that locally Hamiltonian graphs with $\Delta = 6$ are fully cycle extendable.

The Hamilton Cycle Problem for graphs with small maximum degree remains difficult, even when additional structural properties are imposed on the graph. For example, the Hamilton Cycle Problem is NP-complete for bipartite planar graphs with $\Delta \leq 3$ (see [1]), for r-regular graphs for any fixed r (see [18]) and even for planar cubic 3-connected claw-free graphs (see [14]).

Some progress has been made for locally connected graphs with small maximum degree. The first result in this connection was obtained by Chartrand and Pippert [6]. They showed that if G is a connected, locally connected graph of order at least 3 with $\Delta(G) \leq 4$, then G is either Hamiltonian or isomorphic to the complete 3-partite graph $K_{1,1,3}$. Gordon, Orlovich, Potts and Strusevich [10] strengthened their result by showing that apart from $K_{1,1,3}$ all connected, locally connected graphs with maximum degree at most 4 are in fact fully cycle extendable. Since $K_{1,1,3}$ is weakly pancyclic, Ryjáček's conjecture holds for locally connected graphs with maximum degree at most 4.

Global cycle properties of connected, locally connected graphs with maximum degree 5 were investigated in [10,11,13]. Their combined results imply that every connected, locally connected graph with $\Delta = 5$ and $\delta \geq 3$ is fully cycle extendable. This is not the case if $\Delta = 5$ and $\delta = 2$. As mentioned above these graphs are weakly pancyclic, but infinitely many are nonhamiltonian as was shown in [22]. It was also shown in [22] that if 'local connectedness' is replaced by 'local traceability' in graphs with $\Delta = 5$, then apart from three exceptional cases all these graphs are fully cycle extendable. With a stronger local condition, namely 'local Hamiltonicity', it was shown in [22] that an even richer cycle structure is guaranteed. More specifically it was shown that every connected, locally Hamiltonian graph with $\Delta \leq 6$ is fully cycle extendable.

Let G be a locally connected graph with maximum degree Δ. Then for every $v \in V(G)$, $diam(\langle N(v) \rangle) \leq \Delta - 1$. We define a graph G to be *locally k-diameter bounded* if $diam(\langle N(v) \rangle) \leq k$ for all $v \in V(G)$. If, for a given fixed value of Δ, and every k, $1 \leq k \leq \Delta - 1$, it is possible to determine (efficiently) when a locally k-diameter bounded graph is Hamiltonian, then the Hamilton cycle problem of locally connected graphs with maximum degree Δ can be solved (efficiently). We observe that a locally traceable graph with maximum degree Δ is locally $(\Delta - 1)$-diameter bounded and that a locally Hamiltonian graph with maximum degree Δ is $\lfloor \Delta/2 \rfloor$-diameter bounded. These observations motivate the study of the cycle structure of locally k-diameter bounded graphs with 'small' maximum degree.

For $k = 2$, the locally k-diameter bounded graphs have the property that $\langle N(v) \rangle$ is an *isometric* (i.e. a distance preserving) subgraph of G for all $v \in V(G)$. We thus refer to locally 2-diameter bounded graphs as *locally isometric* graphs. In this paper we show that all locally isometric graphs with $\Delta \leq 6$ are weakly pancyclic. For $\Delta \leq 5$ we give complete structural characterizations of those locally isometric graphs that are not fully cycle extendable. We show that every locally isometric graph with $\Delta = 6$ that does not contain a pair of true twins of degree 6 is fully cycle extendable. We conclude by showing that the Hamilton cycle problem for (i) locally 3-diameter bounded graphs having maximum degree 7 and (ii) locally isometric graphs with $\Delta = 8$ is NP-complete.