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On self-clique shoal graphs[☆]

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A B S T R A C T

The *clique graph* of a graph G is the intersection graph $K(G)$ of its (maximal) cliques, and G is *self-clique* if $K(G)$ is isomorphic to G. A graph G is *locally H* if the neighborhood of each vertex is isomorphic to H. Assuming that each clique of the regular and self-clique graph G is a triangle, it is known that G can only be r-regular for $r \in \{4, 5, 6\}$ and G must be, depending on r, a locally H graph for some $H \in \{P_4, P_2 \cup P_3, 3P_2\}$. The self-clique locally P_4 graphs are easy to classify, but only a family of locally H self-clique graphs was known for $H = P_2 \cup P_3$, and another one for $H = 3P_2$.

We study locally $P_2 \cup P_3$ graphs (i.e. *shoal graphs*). We show that all previously known shoal graphs were self-clique. We give a bijection from (finite) shoal graphs to 2-regular digraphs without directed 3-cycles. Under this translation, self-clique graphs correspond to self-dual digraphs, which simplifies constructions, calculations and proofs. We compute the numbers, for each $n \leq 28$, of self-clique and non-self-clique shoal graphs of order n, and also prove that these numbers grow at least exponentially with n.

1. Introduction.

Our graphs are simple and, unless they clearly are not (as, e.g. $P_2 \cup P_3$), also connected. We deal mostly with finite graphs, but some infinite graphs are also considered. We will be explicit about finiteness or infiniteness when needed. A *clique* of a graph G is a maximal complete subgraph of G, or just its set of vertices, as we identify induced subgraphs with their vertex sets. The *clique graph* of G is the intersection graph $K(G)$ of the cliques of G, and G is *self-clique* if G is connected and $K(G) \cong G$. The study of self-clique graphs began in [9] and has been pursued in [1–7,13–16]. A graph is *locally H* if the (open) neighborhood $N(v)$ of any vertex $v \in G$ induces a subgraph isomorphic to H. We denote by P_n the path graph on n vertices and by kP_n the disjoint union of k copies of P_n.

This research was motivated by the paper [7], in which Chia and Ong propose the study of those self-clique graphs whose cliques have all the same size. For $n \geq 2$, they defined $\mathcal{G}(n)$ as the class of all, not necessarily finite, self-clique graphs having only cliques of n vertices. For $n = 2$ they proved that $\mathcal{G}(2)$ only contains the cycles C_n with $n \geq 4$, the one-way infinite path P_∞ and the two-way infinite path (or *infinite cycle*) C_∞. After this, [7] focuses into $\mathcal{G}(3)$, a much tougher proposition. For our

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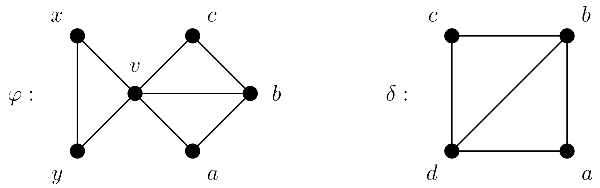

Fig. 2.1. A fish φ and a diamond δ in a shoal graph G.

purposes, their key results [7, Thm. 2, Cor. 1] are that any vertex v of a graph G in $\mathcal{G}(3)$ has, according to its degree, an open neighborhood $N(v)$ which can only be one of the graphs $P_2, P_3, P_4, P_2 \cup P_3$ or $3P_2$, and that any r-regular graph in $\mathcal{G}(3)$ must satisfy $r \in \{4, 5, 6\}$. In particular, a regular graph in $\mathcal{G}(3)$ must be locally H for some $H \in \{P_4, P_2 \cup P_3, 3P_2\}$.

The 4-regular graphs in $\mathcal{G}(3)$, i.e. the locally P_4 graphs in $\mathcal{G}(3)$, were classified in [7]. Hall had shown in [11] that the locally P_4 graphs are just the squared cycles C^2_∞ and C^2_n for all integers $n \geq 7$. Being self-clique, they are all in $\mathcal{G}(3)$, as implied by [7, Thm. 4]. As for the remaining two cases ($r = 5, 6$) of regular graphs in $\mathcal{G}(3)$, only a family of examples was given for each type, and the question was raised in [7, §6] whether they could also be classified.

This work is devoted to the study of 5-regular graphs in $\mathcal{G}(3)$. In other words, we investigate which locally $P_2 \cup P_3$ graphs (to be renamed *shoal graphs* in the next section) are self-clique.

After a preliminary study of locally $P_2 \cup P_3$ graphs in Section 2, we show in Section 3 that their clique graphs are obtained by just flipping the diagonals of all their diamonds.

Hall [11] had proved by examples the existence of locally $P_2 \cup P_3$ graphs, and Chia an Ong's family of self-clique graphs of this type is a proper subfamily of Hall's [7, §3]. We shall prove that all graphs in Hall's family (which as far as we can tell were all the previously known locally $P_2 \cup P_3$ graphs) are indeed self-clique. In fact, Hall's graphs are "orientable", and our geometrical proof also works for the corresponding "non-orientable" analogues, see our Section 4.

In Section 5 we translate our problem into that of finding the self-dual *fishy digraphs* (i.e. balanced orientations of quartic graphs without directed 3-cycles). This greatly simplifies the analysis. The fishy digraph D associated to a locally $P_2 \cup P_3$ graph G has half the number of vertices of G, and its underlying graph is 4-regular, while G is 5-regular. Self-duality for D is simpler than self-cliqueness for G. Quartic graphs have been much more studied than locally $P_2 \cup P_3$ graphs, and there are available catalogs and computer programs to work with them.

Using fishy digraphs we give in Section 6 two new and easy families of locally $P_2 \cup P_3$ graphs, all of them self-clique. We now have examples of each even order greater than 12, which are all possible orders. Up to this point it could conceivably be thought that every locally $P_2 \cup P_3$ graph is self-clique. But our approach also simplified the exhaustive calculation of small examples by hand. This yielded that up to order 18 there are 16 locally $P_2 \cup P_3$ graphs, all of them self-clique, but of order 20 there are 114, and only 60 of them are self-clique. We continued these calculations using a computer and found that up to order 28 there are 3,536,172 locally $P_2 \cup P_3$ graphs and precisely 33,108 of them are self-clique (see Section 7).

We shall prove in Section 8 that the number of self-clique locally $P_2 \cup P_3$ graphs and the number of non-self-clique locally $P_2 \cup P_3$ both grow at least exponentially with the order. Also, that the numbers of self-clique and non-self-clique locally $P_2 \cup P_3$ graphs of order $\aleph_0$ is the cardinality of the continuum. It shall be quite clear that the examples constructed in this paper are just a puny fraction of the self-clique locally $P_2 \cup P_3$ graphs. In our view, these results show that self-clique locally $P_2 \cup P_3$ graphs are unclassifiable. This would solve in the negative the classification problem of 5-regular graphs in $\mathcal{G}(3)$ posed in [7, §6, Question (i)]. But of course a formal proof of unclassifiability would require a formal definition of classifiability.

A vertex v of a graph G (i.e. $v \in G$) is *universal* if v is a neighbor of every other vertex in G. A *cone* is a graph G having a universal vertex, called also an *apex* of G. Whenever we speak of a *diamond in G* we mean an *induced* one. By $X \setminus Y$ we denote difference of sets, while $G - H$ is a graph difference. A *digraph* is an oriented graph: the graph must be simple and each edge has to be oriented in *exactly one* direction, i.e. our digraphs have no loops, parallel arrows or anti-parallel arrows. The *opposite* or *dual* D^{op} of a digraph D has the same vertices as D, but all the arrows reversed: $i \to j$ in D^{op} if, and only if, $j \to i$ in D.

2. Shoals, fishes, heads and tails

In a locally $P_2 \cup P_3$ graph G, the closed neighborhood $N[v]$ of each vertex $v \in G$ induces a subgraph as the graph φ in Fig. 2.1, which we call the *fish* of v. The triangle $\{v, x, y\}$ is the *tail* (of the fish) of v, and the diamond $\{v, a, b, c\}$ is the *head* (of the fish) of v. By dint of using these terms we ended up saying that a locally $P_2 \cup P_3$ graph is a *shoal graph*.

In the next three statements G is a shoal graph, and δ, δ' denote diamonds of G.

Lemma 2.1. *No edge of δ forms a triangle with a vertex $x \in G \setminus \delta$.*

Proof. Let the vertices of δ be labeled a, b, c and d as in Fig. 2.1. If x is adjacent to both a and b, then the (open) neighborhood $N_G(b)$ contains a P_4, a contradiction. By symmetry, only the triangle $\{x, b, d\}$ remains possible, but with it $N_G(b)$ would contain a $K_{1,3}$. $\square$