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Approximation algorithms for minimum (weight) connected k-path vertex cover[☆]

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ABSTRACT

A vertex subset C of a connected graph G is called a connected k-path vertex cover ($CVCP_k$) if every path on k vertices contains at least one vertex from C, and the subgraph of G induced by C is connected. This concept originated in the field of security and supervisory control. This paper studies the minimum (weight) $CVCP_k$ problem. We first show that the minimum weight $CVCP_k$ problem can be solved in time $O(n)$ when the graph is a tree, and can be solved in time $O(rn)$ when the graph is a uni-cyclic graph whose unique cycle has length r, where n is the number of vertices. Making use of the algorithm on trees, we present a k-approximation algorithm for the minimum (cardinality) $CVCP_k$ problem under the assumption that the graph has girth at least k. An example is given showing that performance ratio k is asymptotically tight for our algorithm.

1. Introduction

In this paper, we study approximation algorithms for the minimum (weight) connected k-path vertex cover problem. The length of a path is the number of edges on the path. A path of length $k-1$ has k vertices, which is abbreviated as a *k-path*.

Definition 1.1 (*Minimum Weight Connected k-Path Vertex Cover ($MWCVCP_k$)*)**.** Given a graph $G=(V,E)$ and an integer $k \geq 2$, a vertex subset $C \subseteq V$ is a *k-path vertex cover* (VCP_k) of G if each k-path in G contains at least one vertex of C. If furthermore, the subgraph of G induced by C (denoted as $G[C]$) is connected, then C is called a *connected k-path vertex cover* ($CVCP_k$) of G. Given a weight function w on the vertex set V, the *minimum weight connected k-path vertex cover* ($MWCVCP_k$) problem is to find a $CVCP_k$ of G with the minimum weight.

The VCP_k problem first originated in the field of security protocol design for a wireless sensor network. The topology of a wireless sensor network can be modeled as a graph, in which vertices represent sensors and edges represent communication channels between sensors. In recent years, new security protocols for wireless sensor networks emerge. For example, the *k-generalized Canvas scheme*, which was first proposed by Novotny [14], guarantees data integrity under the assumption that at least one vertex is not captured on each path of length $k-1$. Thus at least one vertex on each k-path must be protected. Since a protected vertex costs more, it is desirable to minimize the number of protected vertices. This is exactly a minimum VCP_k problem.

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In some settings, the costs of installing protected vertices are different, in which case, one wants to find a VCP_k of minimum weight. In some applications, it is desirable that vertices in a VCP_k form a group such that information can be shared in the group. Such a consideration requires the subgraph induced by the VCP_k to be connected. In fact, connectivity requirement is widely considered in communication of networks. For example, in the study of virtual backbone in wireless sensor networks [5], connectivity is a basic assumption. Motivated by these considerations, we will incorporate connectivity into the study of minimum weight VCP_k.

1.1. Related works

The minimum VCP_k problem ($MVCP_k$) was first proposed by Novotny [14] under the background of security protocol design.

For the computation complexity of this problem, Brešar et al. [3] gave a polynomial-time approximation-preserving reduction from the minimum vertex cover problem to $MVCP_k$. So, in view of the result in [6], for any $k \geq 2$, $MVCP_k$ cannot be approximated within a factor of 1.3606 unless $NP \subseteq DTIME(n^{\log\log n})$. Tu *et al.* [16] proved that $MVCP_3$ is NP-hard even for a cubic planar graph of girth 3. The NP-hardness of $MVCP_4$ on a cubic graph was proved by Li and Tu [12].

As a start of the study on special classes of graphs, Brešar et al. [3] gave a linear-time algorithm for $MVCP_k$ on trees. Recently, they [4] gave polynomial time algorithms for the weighted version of this problem on complete graphs, cycles, and trees.

Kardoš et al. [9] presented a randomized approximation algorithm for $MVCP_3$ with an expected performance ratio 23/11. They also presented an exact algorithm for $MVCP_3$ with a running time of $O(1.5171^n)$, where n is the number of vertices. For cubic graphs, Tu et al. proposed a 1.57-approximation algorithm for $MVCP_3$ [16] and a 2-approximation algorithm for $MVCP_4$ [12]. These are works on unweighted VCP_k problems.

For the study of approximation algorithms on the weighted version of the VCP_k problem (namely $MWVCP_k$), Tu et al. gave a 2-approximation for $MWVCP_3$ using a local ratio method in [17], and using a primal–dual method in [18].

Tu [15] also studied the problem from the fixed-parameter point of view, and showed that knowing the optimal value k, $MVCP_3$ can be solved in time $O(2^k \cdot k^{3.376} + n^4 m)$.

The minimum weight VCP_k problem is a special case of the *minimum weight vertex deletion problem* [10,11], the goal of which is to select a vertex set with the minimum weight whose deletion results in a graph satisfying a specific property. In [7], Fujito presented a unified approximation algorithm for the vertex deletion problem with nontrivial and hereditary graph properties, using local ratio method.

The first paper studying the minimum VCP_k problem with a requirement of connectivity is [13], in which Liu et al. gave a PTAS for $MCVCP_k$ in unit disk graphs. Unit disk graph is a model of homogeneous wireless network, in which every vertex corresponds to a point on the plane, two vertices are adjacent if and only if the Euclidean distance between their corresponding points is at most one unit. A basis for the PTAS is a k^2-approximation algorithm for $MCVCP_k$ in a general graph.

The minimum cardinality of a VCP_k is denoted as ψ_k. For the study on the bounds for parameter ψ_k, the readers may refer to [3,2,8].

1.2. Our contributions

In this paper, we study the minimum (weight) VCP_k problem under the requirement that the subgraph induced by the VCP_k is connected. The main contribution of this paper includes the following results.

(i) An efficient algorithm is given to solve the $MWVCVP_k$ problem (the weight version) on trees which runs in time $O(n)$. Notice that taking into account the connectivity requirement, our strategy is completely different from the one in [4] which deals with $MWVCP_k$ (without connectivity requirement) on trees.
(ii) Based on the efficient algorithm for $MWCVCP_k$ on trees, we show that a minimum weight $CVCP_k$ on a uni-cyclic graph can be solved in time $O(rn)$, where r is the length of the unique cycle in G.
(iii) A k-approximation algorithm is given for the $MCVCP_k$ problem (the cardinality version) in a general graph under the assumption that the girth of the graph is at least k, where the girth of a graph is the length of a minimum cycle in the graph. This result reduces the performance ratio of k^2 in [13] by an order for graphs with a girth assumption. An example is given showing that performance ratio k is asymptotically tight for our algorithm. In particular, since any simple graph has girth at least 3, the $MCVCP_3$ problem on any simple graph has a 3-approximation.

The remaining of this paper is organized as follows. Section 2 presents the $O(n)$ algorithm for $MWCVCP_k$ on trees. Section 3 studies $MWCVCP_k$ on uni-cyclic graphs. Section 4 gives the k-approximation algorithm for $MCVCP_k$ on a general graph with girth at least k. An example is given showing the tightness of performance ratio k for our algorithm. Section 5 concludes the paper.

2. $MWCVCP_k$ on trees

In this section, we introduce the algorithm for $MWCVCP_k$ on trees. By the connectedness of a $CVCP_k$, the following observation is obvious.