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Estimation of Laplacian spectra of direct and strong product graphs

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ABSTRACT

Calculating a product of multiple graphs has been studied in mathematics, engineering, computer science, and more recently in network science, particularly in the context of multilayer networks. One of the important questions to be addressed in this area is how to characterize spectral properties of a product graph using those of its factor graphs. While several such characterizations have already been obtained analytically (mostly for adjacency spectra), characterization of Laplacian spectra of *direct product* and *strong product* graphs has remained an open problem. Here we develop practical methods to estimate Laplacian spectra of direct and strong product graphs from spectral properties of their factor graphs using a few heuristic assumptions. Numerical experiments showed that the proposed methods produced reasonable estimation with percentage errors confined within a $\pm 10\%$ range for most eigenvalues.

1. Introduction

Calculating a product of multiple graphs has been studied in several disciplines. In mathematics, multiplication of graphs has been studied with a particular interest in their algebraic properties as matrix operators and their implications for topologies of resulting graphs [13,14,4,6,5]. Graph products also appear in engineering as an efficient way to describe discretized structure of objects in structural mechanics [8,7], and in computer science as a generative model of complex networks [12,11,10]. More recently, graph products have also began to appear in network science, particularly in the context of multilayer networks, where multiplication of graphs is often used as a formal way to describe certain types of multilayer network topologies [3,17,9,15,16]. One of the important questions to be addressed in this area is how to characterize spectral properties of a product graph using those of its factor graphs, especially those of Laplacian matrices[1] because of their high relevance to network structure and dynamics.

Several such spectral characterizations have already been obtained analytically for certain product graphs, but they are mostly for adjacency spectra. Characterization of Laplacian spectra has so far been done only for *Cartesian product* graphs. In the meantime, there are other important forms of graph products, such as *direct product* and *strong product* [5], but characterization of Laplacian spectra of those product graphs has turned out to be quite challenging and has remained an open problem to date.

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[1] In this paper, we consider simple Laplacian matrices of graphs (a.k.a., combinatorial Laplacians), and not normalized Laplacians.

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In this paper, we attempt to address this problem by developing practical, computationally efficient methods to estimate Laplacian spectra of direct and strong product graphs from spectral properties of their factor graphs, using a few heuristic assumptions. We evaluated the effectiveness of our proposed methods through numerical experiments, which demonstrated that they successfully produced reasonable estimation of Laplacian spectra with percentage errors confined within a $\pm 10\%$ range for most eigenvalues.

The rest of the paper is structured as follows: In Section 2 we define three fundamental forms of graph products and describe how they can be represented as operations of adjacency matrices. In Section 3 we summarize spectral properties of product graphs that are already known. In Section 4 we design our new methods to estimate Laplacian spectra of direct and strong product graphs, and then evaluate their effectiveness by numerical experiments. Finally, we conclude this paper with discussions of the limitation of the current work and directions of future research in Section 5.

2. Product graphs

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two simple connected graphs, where V_G (or V_H) and E_G (or E_H) are the sets of nodes and edges of G (or H), respectively. We denote an adjacency matrix of graph X as A_X. We also use I_n to represent an $n \times n$ identity matrix.

We consider operations that create a *product graph* of G and H. We call G and H *factor graphs* of the product. The node set of a product graph will be a Cartesian product of V_G and V_H (i.e., $\{(g, h) \mid g \in V_G,\ h \in V_H\}$). Several graph product operators have been proposed and studied in mathematics, which differ from each other regarding how to connect those nodes in the product graph. In this paper, we focus on the following three fundamental graph products [5]:

Cartesian product: Denoted as $G \square H$. Two nodes (g, h) and (g', h') are connected in $G \square H$ if and only if

$$g = g' \quad \text{and} \quad (h, h') \in E_H, \quad \text{or} \quad (g, g') \in E_G \quad \text{and} \quad h = h'. \tag{1}$$

The adjacency matrix of $G \square H$ is given by

$$\begin{aligned} A_{G \square H} &= A_G \oplus A_H & (2)\\ &= A_G \otimes I_{|V_H|} + I_{|V_G|} \otimes A_H, & (3) \end{aligned}$$

where $\oplus$ and $\otimes$ denote a Kronecker sum and a Kronecker product of matrices, respectively. An example is shown in Fig. 1(a).

Direct (tensor) product: Denoted as $G \times H$. Two nodes (g, h) and (g', h') are connected in $G \times H$ if and only if

$$(g, g') \in E_G \quad \text{and} \quad (h, h') \in E_H. \tag{4}$$

The adjacency matrix of $G \times H$ is given by

$$A_{G \times H} = A_G \otimes A_H. \tag{5}$$

An example is shown in Fig. 1(b).

Strong product: Denoted as $G \boxtimes H$. Two nodes (g, h) and (g', h') are connected in $G \boxtimes H$ if and only if

$$g = g' \quad \text{and} \quad (h, h') \in E_H, \quad \text{or} \quad (g, g') \in E_G \quad \text{and} \quad h = h', \quad \text{or} \quad (g, g') \in E_G \quad \text{and} \quad (h, h') \in E_H. \tag{6}$$

The adjacency matrix of $G \boxtimes H$ is given by

$$\begin{aligned} A_{G \boxtimes H} &= A_G \oplus A_H + A_G \otimes A_H & (7)\\ &= (A_G + I_{|V_G|}) \otimes (A_H + I_{|V_H|}) - I_{|V_G|} \otimes I_{|V_H|}. & (8) \end{aligned}$$

As seen above, a strong product is a sum of Cartesian and direct products. An example is shown in Fig. 1(c).

All of these three graph products are commutative, in the sense that $G * H$ and $H * G$ (where $*$ can be either $\square$, $\times$, or $\boxtimes$) are isomorphic to each other.[2] These operations are also associative.

3. Spectral properties of product graphs

Relationships between spectral properties of a product graph and those of its factor graphs have been known for degree and adjacency spectra for all of the three products, as well as Laplacian spectra for Cartesian product [13,4,7,1]. They are summarized below.

[2] The resulting adjacency matrices will be different, but there is always a permutation of rows/columns to make them identical.

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