Discrete Applied Mathematics 205 (2016) 180–190

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The (1, 2)-step competition graph of a pure local tournament that is not round decomposable

Xinhong Zhang [a], Ruijuan Li [b,*]

[a] Department of Applied Mathematics, Taiyuan University of Science and Technology, 030024 Taiyuan, PR China
[b] School of Mathematical Sciences, Shanxi University, 030006 Taiyuan, PR China

ARTICLE INFO

Article history:
Received 19 January 2015
Received in revised form 3 September 2015
Accepted 16 December 2015
Available online 3 February 2016

Keywords:
Competition graph
(1, 2)-step competition graph
Locally semicomplete digraph
Local tournament
Round-decomposable

ABSTRACT

The competition graph of a digraph was introduced by Cohen in 1968 associated with the study of ecosystems. In 2011, Factor et al. (2011), defined the (1, 2)-step competition graph of a digraph which is a generalization of the competition graph and gave a characterization of the (1, 2)-step competition graph of a tournament. In this paper, we characterize the (1, 2)-step competition graph of a pure local tournament that is not round decomposable.

1. Terminology and introduction

Competition graphs were first introduced by Joel Cohen [4] in connection to a problem in ecology in 1968 and have since been extensively studied. An ecological food web can be modeled by a digraph D, where the vertices of D denote the species of the ecosystem, and there is an arc from vertex x to y if x preys on y. The *competition graph* $C(D)$, of this ecological food web has the same vertex set as D, and xy is an edge in $C(D)$ if there is a vertex z such that (x, z) and (y, z) are arcs in D. That means z is the common prey of x and y. The competition graph can be used to reflect the competition relations among the predators in the food web. Furthermore, the study of competition graphs has been applied widely to the coding, channel assignment in communications, modeling of complex systems arising from study of energy and economic systems, etc. For a comprehensive introduction to competition graphs, see [5,7,12,13]. Recent works in competition graph theory include [9,11]. In 2011, the (1, 2)-step competition graph was defined by Factor and Merz in [6]. In this paper, we characterize the (1, 2)-step competition graphs of the pure local tournaments that are not round decomposable.

It will be assumed that the reader is familiar with the concepts of graphs and digraphs. Other terminology can be found in [3]. In this paper, all digraphs are finite and have no parallel arcs and loops. Let D be a digraph on n vertices. We denote by $V(D)$ and $A(D)$ the vertex set and the arc set respectively. If (x, y) is an arc of D, then we write $x \to y$ and say x dominates y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that *A dominates B*, denoted by $A \to B$. If $A \to B$, but there is no arc from B to A, then we say that *A strictly dominates B*, denoted by $A \mapsto B$. Furthermore, $A \rightsquigarrow B$ denotes the property that there is no arc from B to A. An arc (x, y) of a digraph D is *ordinary* if (y, x) is not in D. A cycle Q of a digraph D is *ordinary* if all arcs of Q are ordinary. If x and y are vertices of D, then the *distance* from x to y in D, denoted $d_D(x, y) = d(x, y)$, is the minimum length of an (x, y)-path, if y is reachable from x, and

* Corresponding author.
E-mail address: ruijuanli@sxu.edu.cn (R. Li).

http://dx.doi.org/10.1016/j.dam.2015.12.013

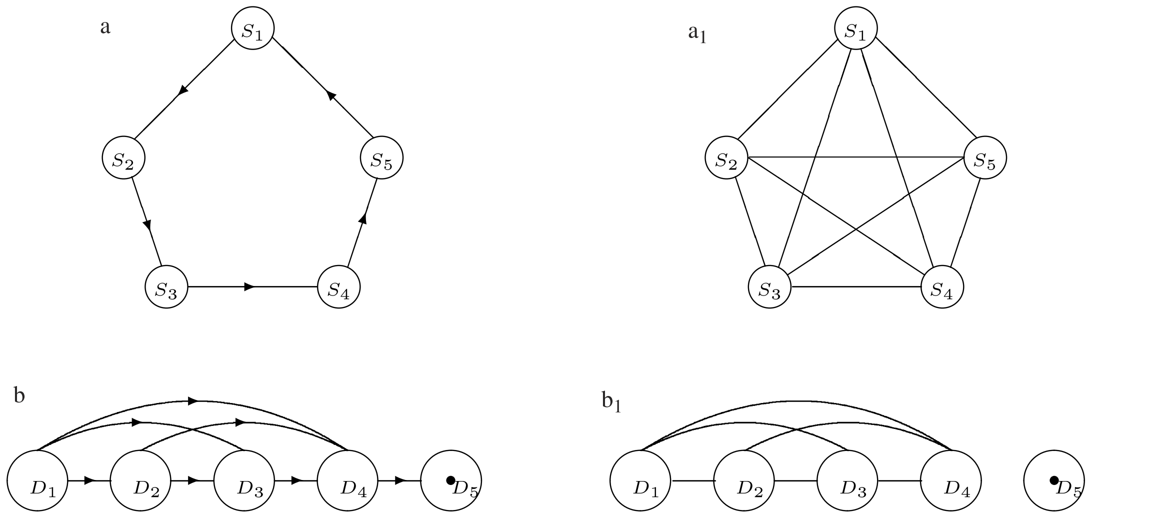

Fig. 1. (a) and (a_1): A strong digraph and its $(1, 2)$-step competition graph, where S_i is a complete subdigraph (subgraph, respectively) with $|V(S_i)| \geq 3$ for $i \in \{1, 2, \ldots, 5\}$; (b) and ($b_1$): A connected digraph which is not strong and its $(1, 2)$-step competition graph, where D_i is a complete subdigraph (subgraph, respectively) with $|V(D_j)| \geq 3$ for $j \in \{1, 2, 3, 4\}$ and $|V(D_5)| = 1$.

otherwise $d(x, y) = \infty$. Furthermore, let H be a subdigraph of D. Then the *distance* from x to H, denoted $d_D(x, H) = d(x, H)$, is the minimum length of an (x, z)-path for all $z \in V(H)$.

The *outset* of a vertex $x \in V(D)$ is the set $N_D^+(x) = \{y|(x, y) \in A(D)\}$. Similarly, $N_D^-(x) = \{y|(y, x) \in A(D)\}$ is the *inset* of x. More generally, for subdigraphs A and B of D, we define the *outset(inset)* of B in A by $N_A^+(B) = \{z|(x, z) \in A(D), x \in V(B), z \in V(A)\}(N_A^-(B) = \{z|(z, x) \in A(D), x \in V(B), z \in V(A)\})$. A subdigraph induced by a subset $U \subseteq V(D)$ is denoted by $D[U]$. In addition, $D - U = D[V(D) \setminus U]$ for any $U \subseteq V(D)$.

A digraph D is *strong* if every vertex of D is reachable by a path from every other vertex of D. For a strong digraph D, a set $S \subseteq V$ is a *separating set* if $D - S$ is not strong. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, D_2, \ldots, D_t$ are the strong components of D, then clearly $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_t) = V(D)$. We call $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_t)$ the *strong decomposition* of D. It is known that the strong components of D can be labeled $D_1, D_2, \ldots, D_t$ such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering* of the strong components of D. The *underlying graph* of D, denoted by $UG(D)$, is the graph obtained by ignoring the orientations of arcs in D and deleting parallel edges. We say that D is *connected* if its underlying graph is connected. In this paper, we only consider connected digraphs.

A digraph D is *semicomplete*, if for any pair of vertices $x, y \in V(D)$, either $(x, y) \in A(D)$, or $(y, x) \in A(D)$, or both. A *tournament* is a semicomplete digraph without a cycle of length 2. A digraph D is *locally semicomplete* (or *a locally semicomplete digraph*), if $D[N_D^+(x)]$ and $D[N_D^-(x)]$ are both semicomplete for every vertex x of D. A locally semicomplete digraph containing no cycle of length 2 is called a *local tournament*. In others words, a local tournament is a locally semicomplete digraph without cycle of length 2. A *pure local tournament* is a local tournament which is not a tournament. Local semicomplete digraphs were introduced in 1990 by Bang-Jensen (see [1]). Local tournaments are the most important subclass of the local semicomplete digraphs. This class of digraphs has many nice properties in common with its subclass, tournaments.

The *(1, 2)-step competition graph* of a digraph D, denoted by $C_{1,2}(D)$, is a graph on $V(D)$ where $xy \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$, such that either $d_{D-y}(x, z) \leq 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) \leq 1$ and $d_{D-y}(x, z) \leq 2$. For example, two digraphs and their respective $(1, 2)$-step competition graphs are shown in Fig. 1.

Let G be a graph. P_i is a path on i vertices in this paper. As for digraphs, a subgraph induced by a subset $X \subseteq V(G)$ is denoted by $G[X]$ and $G - X = G[V(D) \setminus X]$ for $X \subseteq V(D)$. The graph $G - E(H)$ is obtained from G by removing the edges from a subgraph of G that is isomorphic to H.

In [6], Factor et al. completely characterized the $(1, 2)$-step competition graphs of tournaments.

Theorem 1.1 (*Factor & Merz [6]*). *G, a graph on n vertices, is the $(1, 2)$-step competition graph of some tournament if and only if G is one of the following graphs:*

1. *K_n where $n \neq 2, 3, 4$,*
2. *$K_{n-1} \cup K_1$ where $n > 1$,*
3. *$K_n - E(P_3)$ where $n > 2$,*
4. *$K_n - E(P_2)$ where $n \neq 1, 4$, or*
5. *$K_n - E(K_3)$ where $n \geq 3$.*