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Minimum degree distance among cacti with perfect matchings

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ABSTRACT

Let G be a connected graph with vertex set $V(G)$. The degree distance of G is defined as $D'(G) = \sum_{x\in V(G)} d_G(x)D_G(x)$, where $d_G(x)$ is the degree of vertex x, $D_G(x) = \sum_{u\in V(G)} d_G(u, x)$ and $d_G(u, x)$ is the distance between u and x. A connected graph G is called a cactus if any two of its cycles have at most one common vertex. Let $\xi(2n, r)$ be the set of cacti of order $2n$ with a perfect matching and r cycles. In this paper, we give the sharp lower bounds of degree distance among $\xi(2n, r)$ and the corresponding extremal graphs are characterized.

1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $N(v_i) = \{u|uv_i \in E(G)\}$, $N[v_i] = N(v_i) \cup \{v_i\}$. Denote by $d_G(v_i)(= |N(v_i)|)$ the degree of vertex v_i of G. The number $\delta(G) = \min\{d_G(v)|v \in V(G)\}$ is the minimum degree of G, the number $\Delta(G) = \max\{d_G(v)|v \in V(G)\}$ is its maximum degree. For vertices $u, v \in V(G)$, the distance $d_G(u, v)$ is defined as the length of a shortest path between u and v in G. The eccentricity of a vertex x is $ecc_G(x) = max_{y\in V(G)} d_G(x, y)$. The degree distance $D'(G)$ of G, which was introduced by Dobrynin and Kochetova [4] and Gutman [6], is defined as

$$D'(G) = \sum_{x\in V(G)} d_G(x)D_G(x) \tag{1.1}$$

where $d_G(x)$ is the degree of vertex x, $D_G(x) = \sum_{u\in V(G)} d_G(u, x)$. Besides as a topological index itself, the degree distance is also the non-trivial part of the molecular topological index (MTI) (or Schultz index) [12], which may be expressed as $D'(G) + \sum_{u\in V(G)} d_G(u)^2$, for characterization of alkanes [6,10,9]. Some properties for the degree distance may be found, e.g., in [10,9,16] in the text of MTI.

Up to now, many results on the degree distance of graphs are obtained. In [7], Hou and Chang obtained the maximum degree distance among unicyclic graphs on n vertices. Tomescu [13] deduced the minimum degree distance of unicyclic and bicyclic graphs, and the authors in [8] characterized n-vertex unicyclic graphs with girth k and having minimum and maximum degree distance and the maximum degree distance among bicyclic graphs, respectively. In [2] the authors reported the minimum degree distance of graphs with given order and size. In [14], Tomescu presented the graph with minimum degree distance among all connected graphs and disproved a conjecture posed in [4]. In [15] some properties of graphs having minimum degree distance in the class of connected graphs of order n and size $m \geq n - 1$ were deduced.

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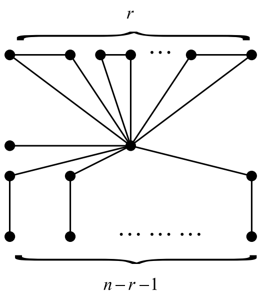

Fig. 1. The graph $G(2n, r)$.

Dankelmann et al. [3] presented an asymptotically sharp upper bound of the degree distance of graphs with given order and diameter.

In this paper, we will further study the degree distance of cacti. We call G a cactus if it is connected and all of blocks of G are either edges or cycles, i.e., any two of its cycles have at most one common vertex. Denote $\xi(2n, r)$ as the set of cacti of order $2n$ with a perfect matching and r cycles and vertex set $\{1, 2, \ldots, 2n\}$. In this paper, we give the sharp lower bounds of degree distance among $\xi(2n, r)$ and the corresponding extremal graphs are characterized.

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1]. If $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them, $G[W]$ the induced subgraph of G. Similarly, if $E \subset E(G)$, we denote by $G - E$ the subgraph of G obtained by deleting the edges of E. If $W = \{v\}$ and $E = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively.

Now we give some lemmas that will be used in the proof of our main results.

Lemma 1.1 ([5]). *Let $G \in \xi(2n, 0)$, where $n \geq 2$. Then G has a pendent vertex whose unique neighbor is of degree two.*

For a connected graph G with $u \in V(G)$, we define $D_G^*(u) = \sum_{x\in V(G)} d_G(x)d_G(x, u)$.

Lemma 1.2 ([8]). *Let G be a graph of order n and v be a pendent vertex of G with $uv \in E(G)$. Then*

$$D'(G) = D'(G - v) + D_{G-v}(u) + D_G(v) + D^*(v).$$

Let G_1, G_2 be two connected graphs, $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$.

Lemma 1.3 ([8]). *Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $G = G_1 \cup G_2$. Let $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$ for $i = 1, 2$. Then*

$$D'(G) = D'(G_1) + D'(G_2) + 2m_1D_{G_2}(v) + 2m_2D_{G_1}(v) + (n_1 - 1)D^*_{G_2}(v) + (n_2 - 1)D^*_{G_1}(v).$$

Lemma 1.4 ([11]). *If $G \in \xi(2n, r)$, $n \geq 3$, then each vertex of G is adjacent to at most one pendent vertex.*

2. The minimum degree distance of cacti with perfect matching

Lemma 2.1. *Let $G(2n, r)$ be the graph as shown in Fig. 1. Then*

$$D'(G(2n, r)) = 20n^2 + 2nr - 2r^2 - 34n + 16.$$

Proof. Let $A = G(2n, r)$. By (1.1),

$$\begin{aligned}
D'(G(2n, r)) &= \sum_{x\in V(A)} d_A(x)D_A(x) \\
&= \sum_{x\in V(A), d_A(x)=1} D_A(x) + (n + r) \sum_{x\in V(A), d_A(x)=n+r} D_A(x) + 2 \sum_{x\in V(A), d_A(x)=2} D_A(x) \\
&= [(5n - r - 4) + (n - r - 1)(7n - r - 8)] + (n + r)(3n - r - 2) \\
&\quad + [(n - r - 1)(10n - 2r - 12) + 2r(10n - 2r - 10)] \\
&= 20n^2 + 2nr - 2r^2 - 34n + 16. \quad \square
\end{aligned}$$