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Averaging 2-rainbow domination and Roman domination

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ABSTRACT

For a graph G, let $\gamma_{r2}(G)$ and $\gamma_R(G)$ denote the 2-rainbow domination number and the Roman domination number, respectively. Fujita and Furuya (2013) proved $\gamma_{r2}(G)+\gamma_R(G) \leq \frac{6}{4}n(G)$ for a connected graph G of order $n(G)$ at least 3. Furthermore, they conjectured $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$ for a connected graph G of minimum degree at least 2 that is distinct from C_5. We characterize all extremal graphs for their inequality and prove their conjecture.

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1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology and notation.

Rainbow domination of graphs was introduced in [1]. Here we consider the special case of 2-rainbow domination. A 2-*rainbow dominating function of a graph* G is a function $f : V(G) \to 2^{\{1,2\}}$ such that $\bigcup_{v\in N_G(u)} f(v) = \{1,2\}$ for every vertex u of G with $f(u) = \emptyset$. The *weight of* f is $\sum_{u\in V(G)} |f(u)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ *of* G is the minimum weight of a 2-rainbow dominating function of G. Roman domination was introduced in [5]. A *Roman dominating function of a graph* G is a function $g : V(G) \to \{0, 1, 2\}$ such that every vertex u of G with $g(u) = 0$ has a neighbor v with $g(v) = 2$. The *weight of* g is $\sum_{u\in V(G)} g(u)$. The *Roman domination number* $\gamma_R(G)$ *of* G is the minimum weight of a Roman dominating function of G.

The definitions of the above two types of dominating functions have some obvious similarities; vertices contribute either 0 or 1 or 2 to the weight of these functions; for vertices that contribute 0, their neighbors contribute at least 2 in total; vertices that contribute 1 do not impose any condition on their neighbors; and vertices that contribute 2 satisfy the requirements of all their neighbors that contribute 0. Nevertheless, while vertices that contribute 1 are useless for their neighbors in Roman domination, they can satisfy 'half' the requirements of their neighbors in 2-rainbow domination.

As observed in [7,3] the two domination parameters are related by the following simple inequalities

$$\gamma_{r2}(G) \leq \gamma_R(G) \leq \frac{3}{2}\gamma_{r2}(G). \tag{1}$$

In fact, if g is a Roman dominating function of a graph G of weight w, then

$$f : V(G) \to 2^{\{1,2\}} : u \mapsto \begin{cases} \emptyset, & \text{if } g(u) = 0, \\ \{1\}, & \text{if } g(u) = 1, \text{ and} \\ \{1,2\}, & \text{if } g(u) = 2 \end{cases}$$

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is a 2-rainbow dominating function of G of weight w, which implies $\gamma_{r2}(G) \leq \gamma_R(G)$. Similarly, if f is a 2-rainbow dominating function of G of weight w and $i \in \{1, 2\}$ is such that $|f^{-1}(\{i\})| \geq |f^{-1}(\{3-i\})|$, then

$$g : V(G) \to \{0, 1, 2\} : u \mapsto \begin{cases} 0, & \text{if } f(u) = \emptyset, \\ 1, & \text{if } f(u) = \{i\}, \text{ and} \\ 2, & \text{if } f(u) \in \{\{3-i\}, \{1, 2\}\} \end{cases}$$

is a Roman dominating function of G of weight at most $3w/2$, which implies $\gamma_R(G) \leq \frac{3}{2}\gamma_{r2}(G)$.

The following result summarizes known tight bounds for the two parameters [6,2,4].

Theorem 1. *Let G be a connected graph of order $n(G)$ at least 3.*

(i) $\gamma_{r2}(G) \leq \frac{3}{4}n(G)$ [6].
(ii) $\gamma_R(G) \leq \frac{4}{5}n(G)$ [2].
(iii) *If G has minimum degree at least 2, then $\gamma_{r2}(G) \leq \frac{2}{3}n(G)$* [4].
(iv) *If G has order at least 9 and minimum degree at least 2, then $\gamma_R(G) \leq \frac{8}{11}n(G)$* [2].

Also bounds on linear combinations of the parameters were considered.

Theorem 2 (*Fujita and Furuya [4]*)**.** *If G is a connected graph of order $n(G)$ at least 3, then $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{6}{4}n(G)$.*

In view of Theorem 1(i) and (ii), one would expect an upper bound on $(\gamma_{r2}(G)+\gamma_R(G))/(2n(G))$ that is somewhere between 3/4 and 4/5. Theorem 2 is slightly surprising as it shows that this upper bound has the smallest possible value, namely 3/4. In fact, (1) and Theorem 2 imply Theorem 1(i).

Our first result is the following.

Theorem 3. *If G is a connected graph of minimum degree at least 2 that is distinct from C_5, then $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$.*

Theorem 3 confirms a conjecture of Fujita and Furuya (Conjecture 2.11 in [4]). Similarly as for Theorem 2, it is again slightly surprising that the upper bound on $(\gamma_{r2}(G)+\gamma_R(G))/(2n(G))$ in Theorem 3 has the smallest of the possible values suggested by Theorem 1(iii) and (iv), namely 2/3. Note that (1) and Theorem 3 imply Theorem 1(iii).

Another result concerning linear combinations of the parameters is the following.

Proposition 4 (*Chellali and Rad [3]*)**.** *There is no constant c such that $2\gamma_{r2}(G)+\gamma_R(G) \leq 2n(G)+c$ for every connected graph G.*

Chellali and Rad posed the problem (Problem 13 in [3]) to find a sharp upper bound on $2\gamma_{r2}(G)+\gamma_R(G)$ for connected graphs G of order at least 3. In fact, (1) and Theorem 2 immediately imply the following.

Corollary 5. *If G is a connected graph of order $n(G)$ at least 3, then $2\gamma_{r2}(G) + \gamma_R(G) \leq \frac{9}{4}n(G)$.*

Corollary 5 is sharp and hence solves the problem posed by Chellali and Rad.

As our second result we characterize all extremal graphs for Theorem 2, all of which are also extremal for Corollary 5.

2. Results and proofs

Our proof of Theorem 3 relies on an elegant approach from [2]. We also use the reductions described in Lemma 4.1 in [2]. Unfortunately, the proofs of (b) and (c) of Lemma 4.1 in [2] are not completely correct; the graphs G' considered in these proofs may have vertices of degree less than 2. We incorporate corrected proofs for these reductions as claims within the proof of Theorem 3.

Lemma 6. *Let G be a graph that contains an induced path P of order 5 whose internal vertices have degree 2. If G' arises from G by contracting three edges of P, then $n(G') = n(G) - 3$, $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$, and $\gamma_R(G) \leq \gamma_R(G') + 2$.*

Proof. Let $P : xuvwy$, that is, G' arises from G by deleting u, v, and w, and adding the edge xy. Clearly, $n(G') = n(G) - 3$.

Let f be a 2-rainbow dominating function of G'. If $f(x), f(y) \neq \emptyset$ or $f(x) = f(y) = \emptyset$, then setting $f(u) = f(w) = \emptyset$ and $f(v) = \{1, 2\}$ extends f to a 2-rainbow dominating function of G. Now we assume that $1 \in f(x)$ and $f(y) = \emptyset$. If $f(x) = \{1\}$, then setting $f(u) = \emptyset, f(v) = \{2\}$, and $f(w) = \{1\}$ extends f to a 2-rainbow dominating function of G. Finally, if $f(x) = \{1, 2\}$, then setting $f(u) = f(v) = \emptyset$ and $f(w) = \{1, 2\}$ extends f to a 2-rainbow dominating function of G. By symmetry, this implies $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$.

Let g be a Roman dominating function of G'. If $g(x) = 2$ and $g(y) = 0$, then setting $g(u) = g(v) = 0$ and $g(w) = 2$ extends g to a Roman dominating function of G. Now we assume that $\{g(x), g(y)\} \neq \{0, 2\}$. Setting $g(u) = g(w) = 0$ and $g(v) = 2$ extends g to a Roman dominating function of G. By symmetry, this implies $\gamma_R(G) \leq \gamma_R(G') + 2$. $\square$

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