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A note on the Grundy number and graph products[☆]

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ABSTRACT

A proper colouring is referred to as a Grundy colouring, or first-fit colouring if every vertex has a neighbour from each of the colour classes lower than its own. The Grundy number of a graph is the maximum k (number of colours) such that a Grundy colouring exists.

In this note, we determine lower and upper bounds for the Grundy number of strong products of graphs, which lead to exact values for the product of some graph classes. We also provide an upper bound on the Grundy number of the strong product of n paths of length 2, which generalizes to an upper bound on the Grundy number of the strong product of n stars.

1. Introduction

A **proper colouring** of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that for any edge $uv \in E, c(u) \neq c(v)$. For each $i = 1, 2, \ldots, k$, let $C_i = \{v \mid c(v) = i\}$. We say that G is k-**colourable** if such a colouring c exists. The **chromatic number** of G, denoted $\chi(G)$, is the minimum k such that a k-colouring exists. We use $u \sim v$ to say u is adjacent to v. We refer to each $i \in \{1, 2, \ldots, k\}$ as a colour and call $C_1, C_2, \ldots, C_k$ the **colour classes** of c and note that each colour class is an independent set.

A proper colouring c is a **Grundy colouring** or a **first-fit colouring** if for every vertex $v \in C_i$, $N(v) \cap C_j \neq \emptyset$ for all $i, j \in \{1, 2, \ldots, k\}$ such that $j < i$. In other words, every vertex has a neighbour from each of the colour classes lower than its own. The **Grundy number** of G, denoted $\Gamma(G)$, is the maximum k such that a Grundy colouring exists. Consider, for example, P_4. Certainly $\chi(P_4) = 2$, however, one can easily determine that $\Gamma(P_4) = 3$ with the labelling given in Fig. 1.

The problem appears to have begun with [8] and the analysis of the game NIM (with every other short impartial game) and we begin with an explanation to the connection to graph products. A short, impartial game can be represented as an acyclic, directed graph, henceforth called the game-graph, with a token on a vertex. The two players take turns moving the token along the edges and the loser is the first player who cannot move the token. The *Grundy-value* (more recently called

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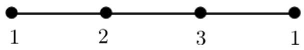

Fig. 1. A proper colouring of P_4 using 3 colours.

the *nimber*) is obtained by the following algorithm: all sinks of the graph receive value 0; the value of vertex x is the least non-negative integer that does not appear as the value of any vertex y where xy is an edge. This numbering gives the winning strategy: always move the token to a vertex labelled 0. For a given game, drawing the game graph and using this algorithm, whilst tedious, actually only needs the outcome classes (Grundy value of 0 means a Previous player win; a non-zero Grundy value is a Next player win). Imagine now playing with several game-graphs, a token on each component and on each turn the player chooses one of the game-graphs and moves the token only on that graph. The loser again is the first player who does not have a legal move, at this point all the tokens will be on sinks. (This is called the disjunctive sum of the games.) The beauty of the Grundy values is that one does not have to draw the game-graph of the new super-game; and at any point in the game for the strategy, the Grundy value of the present position is the exclusive- or binary representation of the Grundy-values of the individual positions. This also shows that the Grundy-value of a vertex $(x_1, x_2, \ldots, x_n)$ in the game-graph of the super-game is no more than twice the maximum Grundy values of the individual vertices.

Note that the graph of the super-game is the Cartesian product of the individual game-graphs—(a, x), (b, y) is an edge if either both $a = b$ and xy is an edge or both ab is an edge and $x = y$. Replacing the directed edges by undirected edges, then the Grundy-value is just one proper-colouring but the disjunctive sum shows that the worst Grundy colouring is no more than twice the value of the worst Grundy colouring of the factors of the Cartesian product. The problem is, of course, to find the worst colouring of the factors.

There are many other super-game compounds (see [5,6]), some of which have graph product interpretations. Of most interest here are:

- Players must move in all components (conjunctive compound) corresponds to the categorical product—(a, x), (b, y) is an edge if both ab and xy are edges.
- Players must move in any non-empty subset of components (selective compound) corresponds to the strong product—(a, x), (b, y) is an edge if ab is an edge and $x = y$; $a = b$ and xy is an edge; or both ab and xy are edges.

The winning strategies are not given in terms of Grundy-values but other functions (specifically remoteness and suspense functions) so the game-theoretic literature [8,5,6,9] contains no heuristics to guide us.

On the computer science side, the Grundy number of a graph represents the worst case colouring that can result from the greedy algorithm. The ratio of the Grundy and chromatic numbers is therefore of interest in complexity arguments. For graphs with 'structure', one would hope that this ratio can be bounded in terms of the 'structure'. In this paper, the structure is induced by the graph product.

The Grundy number of Cartesian and lexicographical products of graphs have been studied in [1,2,4,7]. In this paper, we will examine the **strong products** of (undirected) graphs. The strong product of G and H, denoted $G \boxtimes H$, has vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G \boxtimes H)$ if and only if one of the following holds:

- $u_1u_2 \in E(G)$ and $v_1 = v_2$
- $u_1 = u_2$ and $v_1v_2 \in E(H)$
- $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

The only previously known results regarding the Grundy number of strong products of graphs were studied in [3,4]. In [3], they considered paths and cycles and determined $\Gamma(P_r \boxtimes P_s)$, $\Gamma(P_r \boxtimes C_s)$, and $\Gamma(C_r \boxtimes C_s)$. In [4], they determined that for every $k \geq 3$, there exists a graph G such that $\Gamma(G) = 3$ and $\Gamma(G \boxtimes G) \geq k$. Additionally, the following bound on $\Gamma(G \boxtimes K_n)$ was given for all G such that $\Gamma(G) \geq 2$ and $n \geq 2$:

$$\Gamma(G \boxtimes K_n) \leq (n-1)2^{\Gamma(G)-1} + \Gamma(G). \tag{1}$$

We also provide an upper bound on such a product and compare the two upper bounds at the end of Section 2.

In the following sections, we determine lower and upper bounds for the Grundy number of strong products of graphs, which lead to exact values for products of some graph classes. For the Cartesian product, the hypercubes form the first non-trivial (iterated) product (i.e. P_2 with itself). For the strong product, this is the product of P_3 with itself. Consequently, we provide an upper bound on the Grundy number of the strong product of n paths of length 2, which generalizes to an upper bound on the Grundy number of the strong product of n stars. Graph classes considered include the path on n vertices P_n; the cycle on n vertices C_n; the wheel on $n+1$ vertices W_n; and the complete graph on n vertices K_n.

2. New bounds on $\Gamma(G \boxtimes H)$

A **full** k-**colouring** is a proper colouring c for which every vertex $v \in C_i$ satisfies $N(v) \cap C_j \neq \emptyset$ for all $i, j \in \{1, 2, \ldots, k\}$ such that $i \neq j$. We use the fact that every full colouring is also a Grundy colouring to prove the lower bounds in Lemma 1 and Corollary 2.

Lemma 1. *Suppose G and H are graphs and H has a full k-colouring for some integer k. Then $\Gamma(G \boxtimes H) \geq k\Gamma(G)$.*