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On interval representations of graphs

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ABSTRACT

The interval number $i(G)$ of a graph G is the least integer i such that G is the intersection graph of sets of at most i intervals of the real line. The local track number $l(G)$ is the least integer l such that G is the intersection graph of sets of at most l intervals of the real line and such that two intervals of the same vertex belong to different components of the interval representation. The track number $t(G)$ is the least integer t such that $E(G)$ is the union of t interval graphs. We show that the local track number of a planar graph with girth at least 7 is at most 2. We also answer a question of West and Shmoys in 1984 by showing that the recognition of 2-degenerate planar graphs with maximum degree 5 and interval number at most 2 is NP-complete.

1. Introduction

Let G be a graph. A *d-interval representation* of G is a mapping that assigns a set of at most d intervals of the real line to every vertex of G, such two vertices u and v are adjacent if and only if some interval for u intersects some interval for v. The *interval number* $i(G)$ of a graph G is the least integer d such that G has a d-interval representation. A *component* of a d-interval representation is a maximal subset S of the real line such that every point in S is contained in an interval of the representation. A *d-local representation* of G is a d-interval representation of G with the additional requirement that two intervals for the same vertex belong to distinct components. The *local track number* $l(G)$ has been recently introduced by Knauer and Ueckerdt [9] as the least integer d such that G has a d-local representation. The *track number* $t(G)$ is the least integer t such that there exist t interval graphs G_j on the same vertex set as G that satisfy $E(g) = \bigcup_{1\leqslant j\leqslant t} E(G_j)$. such that $E(G)$ is the union of. If $t(G) \leqslant d$, then a *d-track representation* of G is given by the union of the d (1-interval) representations of the graphs G_j for $1 \leqslant j \leqslant d$. The d real lines supporting the representations of the graphs G_j are called *tracks*. Obviously, we have $i(G) \leqslant l(G) \leqslant t(G)$. We say that G is *d-interval* (resp. *d-local*, *d-track*) if $i(G) \leqslant d$ (resp. $l(G) \leqslant d$, $t(G) \leqslant d$).

For every planar graph G, Scheinerman and West [10] obtained that $i(G) \leqslant 3$ and Gonçalves [5] obtained that $t(G) \leqslant 4$. Both bounds are best possible [10,6].

The *girth* of a graph is the length of a shortest cycle. We denote by $\mathcal{P}_g$ the class of planar graphs with girth at least g (note that $\mathcal{P}_3$ is simply the class of planar graphs). In this paper, we obtain the following results (see Fig. 1).

Theorem 1. *The local track number of a graph in $\mathcal{P}_7$ is at most 2.*

West and Shmoys [11] have shown that recognizing d-interval graphs is NP-complete for every fixed $d \geqslant 2$. Gyárfás and West [7] obtained that recognizing 2-track graphs is NP-complete. This result has been extended by Jiang [8] who proved that recognizing d-track graphs is NP-complete for every fixed $d \geqslant 2$. We generalize these results as follows.

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Graph class	track number	local track number	interval number
$\mathcal{P}_3$	4 UB: [5]	3-4	3 UB: [10]
$\mathcal{P}_4$	4 LB: [6]	3 UB: [7]	3 LB: [10]
$\mathcal{P}_5$	3-4	2-3	2-3
$\mathcal{P}_6$	3 LB: [6] UB: [6]	2-3	2-3
$\mathcal{P}_7$	2-3	2 UB: Th. 1	2
$\mathcal{P}_8$	2-3	2	2
$\mathcal{P}_9$	2-3	2	2
$\mathcal{P}_{10}$	2 UB: [6]	2	2

Fig. 1. Table of results.

Theorem 2. *Let $d \geqslant 2$ be a fixed integer. Given a graph G such that $t(G) \leqslant d$ if and only if $i(G) \leqslant d$, determining whether $t(G) \leqslant d$ is NP-complete, even if G is $(K_4, 2K_3)$-free, alternately orientable, a Meyniel graph, and a string graph.*

Recall that $i(G) \leqslant l(G) \leqslant t(G)$ holds unconditionally. Since Theorem 2 holds for graphs such that $t(G) \leqslant d$ if and only if $i(G) \leqslant d$, it uses only one reduction to show that recognizing d-track graphs, d-local graphs, and d-interval graphs is NP-complete for every fixed $d \geqslant 2$.

West and Shmoys [11] asked in 1984 whether it is NP-complete to determine the interval number of a planar graph. The following result implies that both the interval number and the local track number of a planar graph are NP-complete to determine.

Theorem 3. *Given a 2-degenerate planar graph G with maximum degree 5 such that $l(G) \leqslant 2$ if and only if $i(G) \leqslant 2$, determining whether $l(G) \leqslant 2$ is NP-complete.*

Concerning the track number of planar graphs, Gonçalves and Ochem [6] have proved that recognizing d-track bipartite planar graphs is NP-complete for $d = 2$ and $d = 3$.

2. Preliminaries

A k-vertex is a vertex of degree k. A k-clause is a clause of size k. A pair of adjacent vertices x and y are *true twins* if $N[x] = N[y]$. Non-adjacent vertices x and y are *false twins* if $N(x) = N(y)$. In an interval representation, we say that an interval is *displayed* if some part of this interval intersects no other interval. Similarly, the extremity of an interval is *displayed* if it intersects no other interval. By convention, an interval representation does not contain an interval that does not intersect another interval. A vertex is displayed if it is represented by strictly less intervals than what is allowed by the considered representation, or if one of its interval is displayed. We also say that an interval a *covers* an interval b if b is contained in a. The maximum average degree mad(G) of a graph G is defined by $\text{mad(G)} = \max_{H \subseteq G} \{2|E(H)|/|V(H)|\}$. It is well-known that for every graph $G \in \mathcal{P}_g$, we have $\text{mad(G)} \leqslant 2g/(g-2)$ [2].

3. Proof of Theorem 1

We define a *good representation* as a 2-local representation such that every vertex is displayed. We prove that every graph with girth at least 7 and maximum average degree strictly smaller than $\frac{14}{5}$ admits a good representation. Let G be a hypothetical counter-example to this statement that is minimal for the subgraph order. That is, G has girth at least 7 and maximum average degree strictly smaller than $\frac{14}{5}$, G does not admit a good representation, and every proper subgraph of G admits a good representation. The graph G must be connected, since otherwise we would obtain a good representation of G by gathering the good representations of its connected components.

First, suppose for contradiction that G contains a 1-vertex a adjacent to a vertex b. By minimality, $G \backslash a$ admits a good representation. In this representation, b is represented by at most one interval or one interval of b is displayed. In both cases, we can extend the representation into a good representation of G. In the former case, we add two new intersecting intervals in a new component: one interval for a and one interval for b. In the latter case, we add a new interval for a intersecting only an existing displayed interval of b. This contradicts the fact that G has no good representation, thus G contains no 1-vertex.

A vertex is said *weak* if it is a 2-vertex or a 3-vertex adjacent to a 2-vertex.

Now, suppose for contradiction that G contains an induced path $P = \{v_1, \ldots, v_k\}$ on $k \geqslant 2$ vertices such that every v_i has exactly one neighbor in $G \backslash P$. Consider a good representation of the graph G' obtained from G by deleting the edges contained in P. Every vertex v_i has degree 1 in G' and is thus represented by only one interval. So we can extend the good representation of G' to G by representing the edges of P on a new component, which is a contradiction. The case $k = 2$ means that G does not contain two adjacent 2-vertices. The case $3 \leqslant k \leqslant 5$ means that a 3-vertex in G is adjacent to at most one weak vertex.

Finally, suppose for contradiction that G contains a 4-vertex u_3 adjacent to four weak vertices as shown in Fig. 2(a). The vertex v_1 (resp. w_1, v_5, w_5) exists if and only if v_2 (resp. w_2, v_4, w_4) is a weak 3-vertex. The fact that the girth of G is at least 7 ensures that no two of the vertices depicted in Fig. 2(a) can be identified, that is, Fig. 2(a) describes all the possible