# Anti-forcing numbers of perfect matchings of graphs 

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#### Abstract

We define the anti-forcing number of a perfect matching $M$ of a graph $G$ as the minimal number of edges of $G$ whose deletion results in a subgraph with a unique perfect matching $M$, denoted by $a f(G, M)$. The anti-forcing number of a graph proposed by Vukičević and Trinajstić in Kekule structures of molecular graphs is in fact the minimum anti-forcing number of perfect matchings. For plane bipartite graph $G$ with a perfect matching $M$, we obtain a minimax result: af ( $G, M$ ) equals the maximal number of $M$-alternating cycles of $G$ where any two either are disjoint or intersect only at edges in $M$. For a hexagonal system $H$, we show that the maximum anti-forcing number of $H$ equals the Fries number of $H$. As a consequence, we have that the Fries number of $H$ is between the Clar number of $H$ and twice. Further, some extremal graphs are discussed.


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## 1. Introduction

We only consider finite and simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A perfect matching or 1 -factor $M$ of a graph $G$ is a set of edges of $G$ such that each vertex of $G$ is incident with exactly one edge in $M$.

A Kekulé structure of some molecular graph (for example, benzenoid and fullerene) coincides with a perfect matching of a graph. Randić and Klein [20,14] proposed the innate degree of freedom of a Kekulé structure, i.e. the least number of double bonds can determine this entire Kekulé structure, nowadays it is called the forcing number by Harary et al. [13].

A forcing set $S$ of a perfect matching $M$ of $G$ is a subset of $M$ such that $S$ is contained in no other perfect matchings of $G$. The forcing number of $M$ is the smallest cardinality over all forcing sets of $M$, denoted by $f(G, M)$. An edge of $G$ is called a forcing edge if it is contained in exactly one perfect matching of $G$. The minimum (resp. maximum) forcing number of $G$ is the minimum (resp. maximum) value of forcing numbers of all perfect matchings of $G$, denoted by $f(G)$ (resp. $F(G)$ ). In general to compute the minimum forcing number of a graph with the maximum degree 3 is an NP-complete problem [3].

Let $M$ be a perfect matching of a graph $G$. A cycle $C$ of $G$ is called an $M$-alternating cycle if the edges of $C$ appear alternately in $M$ and $E(G) \backslash M$.

Lemma 1.1 ([2,22]). A subset $S \subseteq M$ is a forcing set of $M$ if and only if each $M$-alternating cycle of $G$ contains at least one edge of $S$.

For planar bipartite graphs, Pachter and Kim obtained the following minimax theorem by using Lucchesi and Younger's result in digraphs [18].

[^0]Theorem 1.2 ([19]). Let $M$ be a perfect matching in a planar bipartite graph $G$. Then $f(G, M)=c(M)$, where $c(M)$ is the maximum number of disjoint $M$-alternating cycles of $G$.

A hexagonal system (or benzenoid) is a 2-connected finite plane graph such that every interior face is a regular hexagon of side length one. It can also be formed by a cycle with its interior in the infinite hexagonal lattice on the plane (graphene). A hexagonal system with a perfect matching is viewed as the carbon-skeleton of a benzenoid hydrocarbon.

Let $H$ be a hexagonal system with a perfect matching $M$. A set of disjoint $M$-alternating hexagons of $H$ is called an $M$ resonant set. A set of $M$-alternating hexagons of $H$ (the intersection is allowed) is called an $M$-alternating set. A maximum resonant set of $H$ over all perfect matchings is a Clar structure or Clar set, and its size is the Clar number of $H$, denoted by $\operatorname{cl}(H)$ (cf. [11]). A Fries set of $H$ is a maximum alternating set of $H$ over all perfect matchings and the Fries number of $H$, denoted by Fries $(H)$, is the size of a Fries set of $H$. Both Clar number and Fries number can measure the stability of polycyclic benzenoid hydrocarbons [6,1].

Theorem 1.3 ([28]). Let $H$ be a hexagonal system. Then $F(H)=c l(H)$.
In this paper we consider the anti-forcing number of a graph, which was previously defined by Vukičević and Trinajstić $[26,27]$ as the smallest number of edges whose removal results in a subgraph with a single perfect matching (see Refs. [5,8,9,15,29,30] for some researches on this topic). By an analogous manner as the forcing number we define the anti-forcing number, denoted by $a f(G, M)$, of a perfect matching $M$ of a graph $G$ as the minimal number of edges not in $M$ whose removal to fix a single perfect matching $M$ of $G$. We can see that the anti-forcing number of a graph $G$ is the minimum anti-forcing number of all perfect matchings of $G$. We also show that the anti-forcing number has a close relation with the forcing number: For any perfect matching $M$ of $G, f(G, M) \leq a f(G, M) \leq(\Delta-1) f(G, M)$, where $\Delta$ denotes the maximum degree of $G$. For a plane bipartite graph $G$, we obtain a minimax result: For any perfect matching $M$ of $G$, the anti-forcing number of $M$ equals the maximal number of $M$-alternating cycles of $G$ any two members of which either are disjoint or intersect only at edges in $M$. For a hexagonal system $H$, we show that the maximum anti-forcing number of $H$ equals the Fries number of $H$. As a consequence, we have that the Fries number of $H$ is between the Clar number of $H$ and twice. Discussions for some extremal graphs about the anti-forcing numbers show the anti-forcing number of a graph $G$ with the maximum degree three can achieve the minimum forcing number or twice.

## 2. Anti-forcing number of perfect matchings

An anti-forcing set $S$ of a graph $G$ is a set of edges of $G$ such that $G-S$ has a unique perfect matching. The smallest cardinality of anti-forcing sets of $G$ is called the anti-forcing number of $G$ and denoted by af $(G)$.

Given a perfect matching $M$ of a graph $G$. If $C$ is an $M$-alternating cycle of $G$, then the symmetric difference $M \oplus C$ is another perfect matching of $G$. Here $C$ may be viewed as its edge-set, and for two sets $A$ and $B, A \oplus B:=(A \cup B) \backslash(A \cap B)$. A subset $S \subseteq E(G) \backslash M$ is called an anti-forcing set of $M$ if $G-S$ has a unique perfect matching, that is, $M$.

Lemma 2.1. A set $S$ of edges of $G$ not in $M$ is an anti-forcing set of $M$ if and only if $S$ contains at least one edge of every $M$ alternating cycle of $G$.

Proof. If $S$ is an anti-forcing set of $M$, then $G-S$ has a unique perfect matching, i.e. $M$. So $G-S$ has no $M$-alternating cycles. Otherwise, if $G-S$ has an $M$-alternating cycle $C$, then the symmetric difference $M \oplus C$ is another perfect matching of $G-S$ different from $M$, a contradiction. Hence each $M$-alternating cycle of $G$ contains at least one edge of $S$. Conversely, suppose that $S$ contains at least one edge of every $M$-alternating cycle of $G$. That is, $G-S$ has no $M$-alternating cycles, so $G-S$ has a unique perfect matching.

The smallest cardinality of anti-forcing sets of $M$ is called the anti-forcing number of $M$ and denoted by $a f(G, M)$. So we have the following relations between the forcing number and anti-forcing number.

Theorem 2.2. Let $G$ be a graph with the maximum degree $\Delta$. For any perfect matching $M$ of $G$, we have

$$
f(G, M) \leq a f(G, M) \leq(\Delta-1) f(G, M)
$$

Proof. Given any anti-forcing set $S$ of $M$. For each edge $e$ in $S$, let $e_{1}$ and $e_{2}$ be the edges in $M$ adjacent to $e$. All such edges $e$ in $S$ are replaced with one of $e_{1}$ and $e_{2}$ to get another set $S^{\prime}$ of edges in $M$. It is obvious that $\left|S^{\prime}\right| \leq|S|$. Further we claim that $S^{\prime}$ is a forcing set of $M$. For any $M$-alternating cycle $C$ of $G$, by Lemma $2.1 C$ must contain an edge $e$ in $S$. Then $C$ must pass through both $e_{1}$ and $e_{2}$. By the definition for $S^{\prime}, C$ contains at least one edge of $S^{\prime}$. So Lemma 1.1 implies that $S^{\prime}$ is a forcing set of $M$. Hence the claim holds. So $f(G, M) \leq\left|S^{\prime}\right| \leq|S|$, and the first inequality is proved.

Now we consider the second inequality. Let $F$ be a minimum forcing set of $M$. Then $f(G, M)=|F|$. For each edge $e$ in $F$, we choose all the edges not in $M$ incident with one end of $e$. All such edges form a set $F^{\prime}$ of size no larger than $(\Delta-1)|F|$, which is disjoint with $M$. We claim that $F^{\prime}$ is an anti-forcing set of $M$. Otherwise, Lemma 2.1 implies that $G-F^{\prime}$ contains an $M$-alternating cycle $C$. Since each edge in $F$ is a pendant edge of $G-F^{\prime}, C$ does not pass through an edge of $F$. This contradicts that $F$ is a forcing set of $M$ by Lemma 1.1. Hence af $(G, M) \leq\left|F^{\prime}\right| \leq(\Delta-1)|F|$.

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