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Quasi-centers and radius related to some iterated line digraphs, proofs of several conjectures on de Bruijn and Kautz graphs

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ABSTRACT

Bond (1987) and Bond et al. (1987), conjectured that a quasi-center in an undirected de Bruijn graph $UB(d, D)$ has cardinality at least $d-1$, and that a quasi-center in an undirected Kautz graph $UK(d, D)$ has cardinality at least d. They proved that for $d \geq 3$, the radii of $UB(d, D)$ and $UK(d, D)$ are both equals to D, and conjectured also that the radii of $UB(2, D)$ and $UK(2, D)$ are respectively $D-1$ and D. In this paper we give results in a more general context which validate these conjectures (excepting that asserting that the radius of $UB(2, D)$ is $D-1$), and give simplified proofs of the cited results.

1. Introduction, notation

Let G be a connected graph. The *distance* $d(x, y)$ between two vertices x and y of G is the length of a shortest path between them. For a set S of vertices of G and a vertex x of G, $d(S, x)$ is the minimum of the distances $d(y, x)$ with $y \in S$. The *eccentricity* $e(x)$ of x is the maximum of the distances $d(x, y)$ where y belongs to the vertex set $V(G)$ of G. The *diameter* $D(G)$ of G is the maximum of the distances $d(x, y)$ with x, y in $V(G)$, and it is also the maximum of the eccentricities of the vertices of G. The *radius* $R(G)$ of G is the minimum of the eccentricities. A set S of vertices of G is called a *quasi-center* if for every $x \in V(G)$ we have $d(S, x) < D(G)$.

For a vertex x of a undirected graph G, a vertex y such that $\{x, y\}$ is an edge of G is a *neighbor* of x. The *degree* $d_G(x)$ of x is the number of the neighbors of x.

In this paper, we allow loops in digraphs. For a vertex x of a digraph G, a vertex y such that (x, y) is an arc of G is an *out-neighbor* of x. The *out-degree* $d_G^+(x)$ of x is the number of the out-neighbors of x. A vertex z such that (z, x) is an arc of G is an *in-neighbor* of x. The *in-degree* $d_G^-(x)$ of x is the number of the in-neighbors of x.

In an undirected graph G, a *walk* of length m is a sequence $X_1, \dots, X_{m+1}$ of vertices of G such that X_{i+1} is a neighbor of X_i for $1 \leq i \leq m$. When $X_{m+1} = X_1$, the sequence is called a *closed walk*. A walk with distinct vertices is a *path*, and a closed walk with distinct vertices is a *cycle*.

A *directed walk* of length m, in a digraph G, is a sequence $X_1, \dots, X_{m+1}$ of vertices of G such that X_{i+1} is an out-neighbor of X_i for $1 \leq i \leq m$. When $X_{m+1} = X_1$, the sequence is called a *directed closed walk*. A directed walk with distinct vertices is a *directed path*, and a directed closed walk with distinct vertices is a *directed cycle*. From now on, the sequence $X_1, \dots, X_{m+1}$ will be denoted by $X_1 \dots X_{m+1}$.

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For a digraph G, the *underlying graph* UG of G is the undirected graph obtained from G by removing all the orientations of G (loops included). The notation which follows is that of [5].

An L-walk of length m of UG is a directed walk of length m of G. An R-walk of length m of UG is a walk $X_1 \ldots X_{m+1}$ of UG such that $X_{m+1} \ldots X_1$ is a directed walk of length m of G. An LR-walk of length m is a walk $X_1 \ldots X_{m+1}$ such that there exists i, $1 < i < m+1$ such that $X_1 \ldots X_i$ is an L-walk and $X_i \ldots X_{m+1}$ is an R-walk. Similarly we define an RL-walk.

In a strongly connected digraph G, the distance $d(x, y)$ from the vertex x to the vertex y of G is the length of a shortest directed path from x to y. The diameter $D(G)$ of G is the maximum of the distances $d(x, y)$ with x, y in $V(G)$.

For a digraph G, the *line digraph* $L(G)$ of G, is the digraph whose vertex set is the set $\mathcal{A}(G)$ of the arcs of G, and whose arcs are the couples (xy, yz), where xy and yz are arcs of G. Clearly, for every arc xy of G, we have $d^+_{L(G)}(xy) = d^+_G(y)$ and $d^-_{L(G)}(xy) = d^-_G(x)$. For an integer $n \geq 1$, the nth *iterated line digraph* is the digraph $L^n(G)$ of G, recursively defined by $L^1(G) = L(G)$, and $L^n(G) = L(L^{n-1}(G))$. For convenience, we put $L^0(G) = G$. $L^n(G)$ is also the digraph whose vertices are the directed walks of G of length n, and whose arcs are the ordered pairs $(x_1 \ldots x_{n+1}, y_1 \ldots y_{n+1})$ of directed walks of length n, with $x_2 \ldots x_{n+1} = y_1 \ldots y_n$. It is known that if G is a digraph of diameter D, distinct from a directed cycle, the diameter of the digraph $L^n(G)$ is $D + n$.

For an integer $d \geq 2$, $\mathbb{Z}_d = \{0, \ldots, d-1\}$ is the set of the integers modulo d. For $d \geq 2$ and $D \geq 2$, the *de Bruijn digraph* $B(d, D)$ is the digraph whose vertex set is $\mathbb{Z}_d^D$, and whose arcs are the couples $(x_1 x_2 \ldots x_D, x_2 \ldots x_D i)$ with $i \in \mathbb{Z}_d$. The de Bruijn digraph $B(d, 1)$ is the complete digraph $\vec{K}_d$ (with a loop at each vertex). It is known and easy to prove that $B(d, D)$ is a strongly connected regular digraph of degree d. The *de Bruijn graph* $UB(d, D)$ is the underlying graph of $B(d, D)$. It is known and easy to prove that for $D \geq 2$, the de Bruijn digraph $B(d, D)$ is the line digraph of $B(d, D-1)$, and thus $B(d, D)$ is the $(D-1)$th iterated line digraph of $\vec{K}_d$. It is also known that the diameters of $B(d, D)$ and of $UB(d, D)$ are both equal to D.

For $d \geq 2$ and $D \geq 2$, the *Kautz digraph* $K(d, D)$ is the sub-digraph of $B(d+1, D)$ induced by the set $\Omega(d, D)$ of the vertices $x_1 \ldots x_D$ of $B(d+1, D)$ verifying $x_i \neq x_{i+1}$ for $1 \leq i \leq D-1$. The Kautz digraph $K(d, 1)$ is the complete digraph $\vec{K}^*_{d+1}$ (without loops). It is known and easy to see that $K(d, D)$ is a strongly connected regular digraph of degree d (without loops). The *Kautz graph* $UK(d, D)$ is the underlying graph of $K(d, D)$. It is known and easy to prove that for $D \geq 2$, the Kautz digraph $K(d, D)$ is the line digraph of $K(d, D-1)$, and then $K(d, D)$ is the $(D-1)$th iterated line digraph of $\vec{K}^*_{d+1}$. It is also known that the diameters of $K(d, D)$ and of $UK(d, D)$ are both equal to D.

Note that any set of cardinality d could play the role of $\mathbb{Z}_d$. We introduce now a generalization of de Bruijn and Kautz digraphs.

For $d \geq 2$ and a subset A of $\mathbb{Z}_d$, $G(d, A)$ is the digraph whose vertex set is $\mathbb{Z}_d$, and whose arcs are the ordered pairs (x, y), $x, y \in \mathbb{Z}_d$ and $x \neq y$, and the loops (x, x), $x \in A$. It is clear that the vertices of $G(d, A)$ which are not in A have out-degree and in-degree both equal to $d-1$, and that the vertices of A have out-degree and in-degree both equal to d. It is also clear that $G(d, A)$ is strongly connected, and that the diameters of $G(d, A)$ and $UG(d, A)$ are both equal to 1.

For $D \geq 1$, $G(d, A, D)$ denotes the iterated line digraph $L^{D-1}(G(d, A))$. It is easy to see that when $A = \mathbb{Z}_d$, $G(d, A, D)$ is the de Bruijn digraph $B(d, D)$ (and so $G(d, \mathbb{Z}_d, D) = B(d, D)$), and that when $d \geq 3$ and $A = \emptyset$, $G(d, A, D)$ is the Kautz digraph $K(d-1, D)$ (and so $G(d, \emptyset, D) = K(d-1, D)$).

J. Bond in [1], and J. Bond et al. in [2], claimed the following conjectures:

Conjecture 1.1. *Every quasi-center of the de Bruijn graph $UB(d, D)$ has cardinality at least $d-1$.*

Conjecture 1.2. *Every quasi-center of the Kautz graph $UK(d, D)$ has cardinality at least d.*

Conjecture 1.3. *The radius of the Kautz graph $UK(2, D)$ is D.*

They proved:

Theorem 1.4. *For $d \geq 3$, the radii of $UB(d, D)$ and $UK(d, D)$ are both equal to D.*

In this paper, we prove that the cardinality of a quasi-center of a graph $UG(d, A, D)$ is at least $d-1$, which validates Conjectures 1.1 and 1.2. By using this result, we easily prove that for $d \geq 3$, the radius of a graph $UG(d, A, D)$ is D, which validates Conjecture 1.3, and yields an easier proof of Theorem 1.4.

2. Preliminary results

In [3], the author of this paper defined from a de Bruijn digraph $B(d, D)$, $D \geq 2$ three digraphs, each isomorphic to $B(d, D-1)$. Here, we partially generalize these constructions. More precisely, consider an arbitrary digraph G. Let $\mathcal{R}_1$, be the relation defined on $V(L(G))$ by $x_1 x_2 \, \mathcal{R}_1 \, y_1 y_2 \Leftrightarrow x_2 = y_2$.

It is easy to see that $\mathcal{R}_1$ is an equivalence relation, and that the class of a vertex $X = x_1 x_2$ of $L(G)$ is $C(X) = \{ix_2;\ ix_2 \in \mathcal{A}(G)\}$. We denote by A_1 the set of the equivalence classes, and then G_1 is the digraph whose vertex set is A_1, and whose arcs are the ordered pairs of classes (C, C') such that there exists a vertex $a \in C$ having an out-neighbor $a' \in C'$. We observe that in this case, all the vertices of C have a' as out-neighbor. When the minimum in-degree of G is at least 1, we claim that the map f_1 from $V(G)$ into A_1 defined by $f_1(x) = C(ix)$, where $i \in V(G)$ is an in-neighbor of x, is an isomorphism from G to G_1.