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On star-critical and upper size Ramsey numbers

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ABSTRACT

In this paper, we study the upper size Ramsey number $u(G_1, G_2)$, defined by Erdős and Faudree, as well as the star-critical Ramsey number $r_*(G_1, G_2)$, defined by Hook and Isaak. We define Ramsey-full graphs and size Ramsey good graphs, and perform a detailed study on these graphs. We generalize earlier results by determining $u(nK_k, mK_l)$ and $r_*(nK_k, mK_l)$ for $k, l \geq 3$ and large m, n; $u(C_n, C_m)$ for m odd, with $n > m \geq 3$; and $r_*(C_n, C_m)$ for m odd, with $n \geq m \geq 3$ and $(m, n) \neq (3, 3)$.

1. Introduction

All graphs considered in this paper are finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree, the maximum degree, the chromatic number, the connectivity, the length of a shortest cycle and the length of a longest cycle in G are denoted by $\delta(G)$, $\Delta(G)$, $\chi(G)$, $\kappa(G)$, $g(G)$ and $c(G)$, respectively. For $X \subseteq V(G)$ and $v \in V(G)$, let $G[X]$, $G - X$ and $G - v$ denote the subgraphs induced by X, $V(G) - X$ and $V(G) - \{v\}$, respectively. For a subgraph H of G, let $N_H(v)$ be the set of neighbors of $v \in V(G)$ that is contained in $V(H)$, and let $d_H(v) = |N_H(v)|$. For two disjoint subgraphs H_1 and H_2 of G, we define $H_1 \cup H_2$ as the disjoint union of H_1 and H_2. We use mH to denote m vertex disjoint copies of H. A cycle of order n is denoted by C_n, and a wheel W_n is a graph obtained from a C_n and an additional vertex v by joining v to every vertex of the C_n. A fan F_n is a graph of order $2n + 1$, which is n triangles sharing exactly one vertex. A graph $K_n - e$ is a complete graph K_n with one arbitrary edge deleted. Let $K_{r-1} \sqcup K_{1,k}$ be the graph obtained from a K_{r-1} and an additional vertex v by joining v to k vertices of K_{r-1}. A graph G is pancyclic if it contains cycles of every length between 3 and n, and it is weakly pancyclic if it contains cycles of every length between $g(G)$ and $c(G)$.

To study various graph properties in graph Ramsey theory, we adopt the following definitions and notation.

Definition 1. Given two graphs G_1 and G_2, we say that a graph G arrows the pair (G_1, G_2), denoted by $G \rightarrow (G_1, G_2)$, if in any red–blue coloring of the edges of G, there is a red copy of G_1 or a blue copy of G_2.

For two given graphs G_1 and G_2, the most extensively investigated concept within Ramsey theory is the graph Ramsey number $R(G_1, G_2)$, which is the smallest integer r such that, for any graph G of order r, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. For simplicity, we now restate this definition of $R(G_1, G_2)$ in the language of arrowing.

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Definition 2.

$$r = R(G_1, G_2) = \min\{n \mid K_n \to (G_1, G_2)\}.$$

Let r denote the Ramsey number $R(G_1, G_2)$ throughout the paper. A dynamic survey on Ramsey numbers can be found in [22]. We see that the Ramsey number $R(G_1, G_2)$ is the smallest number of vertices in a graph G such that $G \to (G_1, G_2)$. Analogously, the size Ramsey number $\hat{r}(G_1, G_2)$ is the smallest number of edges in a graph G such that $G \to (G_1, G_2)$, a definition that was introduced by Erdős et al. [11] and studied further extensively by many others.

Definition 3 (*Erdős et al. [11]*).

$$\hat{r}(G_1, G_2) = \min\{|E(G)| \mid G \to (G_1, G_2)\}.$$

For a survey of results on size Ramsey numbers we refer the reader to [15]. In the context of $G \to (G_1, G_2)$, some other graph parameters of G were introduced by Burr et al. [6] in 1976, including the minimum degree of G, a parameter that attracted considerable attention again since 2006.

Definition 4 (*Burr et al. [6]*).

$$s(G_1, G_2) = \min\{\delta(G) \mid G \to (G_1, G_2)\}.$$

To investigate the Ramsey properties of subgraphs of K_r when $r = R(G_1, G_2)$, several other definitions were introduced. In 1991, Erdős and Faudree [9] considered two related definitions: the upper size Ramsey number $u(G_1, G_2)$ and the lower size Ramsey number $\ell(G_1, G_2)$. The latter was considered earlier by Faudree and Sheehan [16] under the term *restricted size Ramsey number*.

Definition 5 (*Erdős and Faudree [9]*).

$$u(G_1, G_2) = \min\{q \mid \text{if } G \subseteq K_r \text{ and } E(G) \geq q, \text{ then } G \to (G_1, G_2)\}.$$

Definition 6 (*Erdős and Faudree [9]*).

$$\ell(G_1, G_2) = \min\{|E(G)| \mid G \subseteq K_r \text{ and } G \to (G_1, G_2)\}.$$

Clearly, $\hat{r}(G_1, G_2) \leq \ell(G_1, G_2) \leq u(G_1, G_2) \leq \binom{r}{2}$. Moreover, if $\ell(G_1, G_2) \leq q < u(G_1, G_2)$, then there exist two graphs $H_1, H_2 \subseteq K_r$ with $|E(H_1)| = |E(H_2)| = q$ such that $H_1 \to (G_1, G_2)$, but $H_2 \nrightarrow (G_1, G_2)$. Erdős and Faudree performed a detailed investigation of the two functions $u(G_1, G_2)$ and $\ell(G_1, G_2)$, and determined some exact values, and some reasonable upper and lower bounds. With respect to general lower bounds, they established the following theorem. There are example graphs showing that these bounds cannot be improved in general.

Theorem 1 (*Erdős and Faudree [9]*). *For any pair of graphs G_1 and G_2 without isolated vertices,*

$$u(G_1, G_2) \geq \binom{r-1}{2} + \delta(G_1) + \delta(G_2) - 1,$$
$$\ell(G_1, G_2) \geq |E(G_1)| + |E(G_2)| - 1.$$

Since $K_r \to (G_1, G_2)$, but $K_{r-1} \nrightarrow (G_1, G_2)$, a natural problem is to consider G such that $K_{r-1} \subseteq G \subseteq K_r$ and $G \to (G_1, G_2)$. To study this, Hook and Isaak [17] introduced the definition of the star-critical Ramsey number $r_*(G_1, G_2)$.

Definition 7 (*Hook and Isaak [17]*).

$$r_*(G_1, G_2) = \min\{\delta(G) \mid G \subseteq K_r \text{ and } G \to (G_1, G_2)\}.$$

There are some equivalent versions of the definition. For example,

$$r_*(G_1, G_2) = \min\{k \mid K_{r-1} \sqcup K_{1,k} \to (G_1, G_2)\}.$$

Recall that $K_{r-1} \sqcup K_{1,k}$ is the graph obtained from K_{r-1} and an additional vertex v by joining v to k vertices of K_{r-1}. If $K_{r-1} \sqcup K_{1,k} \to (G_1, G_2)$, by Definition 7, $r_*(G_1, G_2) \leq k$. On the other hand, if $K_{r-1} \sqcup K_{1,k} \nrightarrow (G_1, G_2)$, then any graph $G \subseteq K_r$ with $\delta(G) \leq k$ is a subgraph of $K_{r-1} \sqcup K_{1,k}$, and hence $G \nrightarrow (G_1, G_2)$. Thus, $r_*(G_1, G_2) \geq k + 1$.

Another equivalent definition is as follows.

$$r_*(G_1, G_2) = \min\{\delta(G) \mid K_{r-1} \subseteq G \subseteq K_r \text{ and } G \to (G_1, G_2)\}.$$

The reason is that, for any graph $G \subseteq K_r$ and $G \to (G_1, G_2)$, there exists a graph G' such that $K_{r-1} \subseteq G' \subseteq K_r$, $G' \to (G_1, G_2)$ and $\delta(G) = \delta(G')$.