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Note

Proofs of conjectures on the Randić index and average eccentricity[☆]

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ABSTRACT

The Randić index $R(G)$ of a graph G is defined by $R(G) = \sum_{uv} \frac{1}{\sqrt{d(u)d(v)}}$, where $d(u)$ is the degree of a vertex u and the summation extends over all edges uv of G. The eccentricity $\epsilon(v)$ of a vertex v is the maximum distance from it to any other vertex and the average eccentricity $\bar{\epsilon}(G)$ of graph G is the mean value of eccentricities of all vertices of G. There are relations between the Randić index and the average eccentricity of connected graphs conjectured by a computer program called AGX: for any connected graph G on $n \geq 14$ vertices, both lower bounds of $R(G) + \bar{\epsilon}(G)$ and $R(G) \cdot \bar{\epsilon}(G)$ are achieved only by a star. In this paper, we show that both conjectures are true.

1. Introduction

The Randić index $R = R(G)$ of a graph G is defined as follows:

$$R = R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ denotes the degree of a vertex u and the summation runs over all edges uv of G. This topological index was first proposed by Randić [24] in 1975, suitable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. It is well correlated with a variety of physico-chemical properties of alkanes. And it is one of the most popular molecular descriptors to which three books [13,14,16] are devoted.

In this paper, we only consider finite, undirected and simple graphs. The degree $d(u)$ of a vertex u is the number of edges incident to it. The distance between two vertices u and v in graph G, denoted by $d_G(u, v)$ (or $d(u, v)$ for short), is the length of a shortest path connecting u and v in G. The eccentricity $\epsilon(v)$ of a vertex v is the maximum distance from it to any other vertex and the average eccentricity $\bar{\epsilon}(G)$ of a graph G is the mean value of eccentricities of all vertices of G. For undefined terminology and notations we refer the reader to the book of Bondy and Murty [6].

There are many results on the relations between the Randć index and some other graph invariants, many of which were conjectured by a computer program called AGX (see [1,3,4,7]), such as the minimum degree [2,8,18], chromatic number [12,20], radius [21,9], diameter [25,22], eccentricity [10], girth [17,23] and so on (see the survey of Li and Shi [19]). There are also some interesting new aspects of Randić index can be found in the recent paper [11]. Recently, Liang and Liu [15] solved two conjectures concerning the relations between the Randić index and average eccentricity, i.e., for any connected graph

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G, both upper bounds of $R(G) + \bar{\epsilon}(G)$ and $R(G) \cdot \bar{\epsilon}(G)$ are achieved only by a path. However, two conjectures that concerning the lower bounds of $R(G) + \bar{\epsilon}(G)$ and $R(G) \cdot \bar{\epsilon}(G)$ are still open.

Conjecture 1.1 (*Conjecture A. 462-L in [1]*). *For any connected graph G on $n \geq 7$ vertices with the Randić index $R(G)$ and the average eccentricity $\bar{\epsilon}(G)$, we have*

$$R(G) + \bar{\epsilon}(G) \geq \sqrt{n-1} + 2 - \frac{1}{n},$$

with equality if and only if G is a star.

Conjecture 1.2 (*Conjecture A. 464-L in [1]*). *For any connected n-vertex graph G with the Randić index $R(G)$ and the average eccentricity $\bar{\epsilon}(G)$, we have*

$$R(G) \cdot \bar{\epsilon}(G) \geq \begin{cases} \frac{n}{2} & \text{if } 3 \leq n \leq 13 \\ \sqrt{n-1}\left(2 - \frac{1}{n}\right) & \text{if } n > 13 \end{cases}$$

with equality if and only if $G = K_n$ if $3 \leq n \leq 13$ or $G = S_n$ if $n > 13$.

In this work, we show that both conjectures are true. The rest of this paper is organized as follows. In Section 2, we give some results which will be used for the proofs of our main results. In Section 3, we give proofs of Conjectures 1.1 and 1.2.

2. Some lemmas

In this section, we give some results that will be needed in the sequel.

In 1998, Bollobás and Erdős [5] gave a result on the minimum Randić index among all n-vertex connected graphs.

Lemma 2.1 (*[5]*). *Let G be a connected n-vertex graph, then*

$$R(G) \geq \sqrt{n-1},$$

the equality holds if and only if G is a star.

In 2002, Delorme et al. [8] gave a result of the minimum Randić index among all n-vertex connected graphs with the minimum degree at least two.

Lemma 2.2 (*[8]*). *For any connected n-vertex graph G with minimum degree $\delta \geq 2$, we have*

$$R(G) \geq \frac{2(n-2)}{\sqrt{2(n-1)}} + \frac{1}{n-1}$$

and the bound is tight if and only if $G = K^{\star}_{2,n-2}$ which arises from complete bipartite graph $K_{2,n-2}$ by joining the vertices in the partite set with two vertices by a new edge.

We give a result concerning the lower bound of the Randić index with given number of vertices of degree $n-1$.

Lemma 2.3. *For any connected n-vertex graph G containing k vertices of degree $n-1$, we have*

$$R(G) \geq \frac{k \cdot (n-k)}{\sqrt{k(n-1)}} + \frac{k(k-1)}{2(n-1)}.$$

Proof. Denote by n_i the number of vertices of degree i and by $x_{i,j}$ the number of edges joining the vertices of degrees i and j in G, then the minimum degree of G is at least k since G contains k vertices of degree $n-1$. The mathematical description of the problem is as follows:

$$\min\ R(G) = \min \sum_{k \leq i \leq j \leq n-1} \frac{x_{i,j}}{\sqrt{ij}}$$

subject to:

$$\sum_{\substack{j=k \\ j \neq i}}^{n-1} x_{i,j} + 2x_{i,i} = in_i \quad \text{for } k \leq i \leq n-1;$$

$$n_k + n_{k+1} + \cdots + n_{n-1} = n;$$