Discrete Applied Mathematics 189 (2015) 41–48

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Simple PTAS's for families of graphs excluding a minor[☆]

Sergio Cabello[a,b,*], David Gajser[a]

[a] Department of Mathematics, IMFM, Slovenia

[b] Department of Mathematics, FMF, University of Ljubljana, Slovenia

ARTICLE INFO

Article history:
Received 21 October 2014
Accepted 4 March 2015
Available online 21 April 2015

Keywords:
Polynomial-time approximation scheme
Forbidden minor
Separator
Local optimization
Maximum independent set
Minimum vertex cover
Minimum dominating set

ABSTRACT

We show that very simple algorithms based on local search are polynomial-time approximation schemes for Maximum Independent Set, Minimum Vertex Cover and Minimum Dominating Set, when the input graphs have a fixed forbidden minor.

1. Introduction

In this paper we present very simple PTAS's (polynomial-time approximation schemes) based on greedy local optimization for Maximum Independent Set, Minimum Vertex Cover and Minimum Dominating Set in minor-free families of graphs. The existence of PTAS's for such problems was shown by Grohe [10], and better time bounds were obtained using the framework of bidimensionality; see the survey [7] and references therein. The advantage of our algorithms is that they are surprisingly simple and do not rely on deep structural results for minor-free families.

A graph H is a *minor* of G if H can be obtained from a subgraph of G by edge contractions. We say that G is *H-minor-free* if H is not its minor. A family of graphs is *H-minor-free* if all the graphs in the family are H-minor-free. It is well-known that the family of planar graphs is $K_{3,3}$-minor-free and K_5-minor-free, and similar results hold for graphs on surfaces. Thus, minor-free families is a vast extension of the family of planar graphs and, more generally, graphs on surfaces. We will restrict our attention to K_h-minor-free graphs, where K_h is the complete graph on h vertices, because H-minor-free graphs are also $K_{|V(H)|}$-minor-free.

The development of PTAS's for graphs with a forbidden fixed minor is often based on a complicated theorem of Robertson and Seymour [15] describing the structure of such graphs. In fact, one needs an algorithmic version of the structural theorem and much work has been done to obtain simpler and faster algorithms finding the decomposition. See Grohe, Kawarabayashi and Reed [11] for the latest improvement and a discussion of previous work. Even those simplifications are still very complicated and, in fact, the description of the structure of K_h-minor-free graphs is cumbersome in itself. Obtaining a PTAS for Maximum Independent Set restricted to minor-free families is easier and can be done through the computation of separators, as shown by Alon, Seymour, and Thomas [2]. However, the approach does not work for Minimum

☆ This work is partially funded by the Slovenian Research Agency.

* Corresponding author at: Department of Mathematics, IMFM, Slovenia.
E-mail addresses: sergio.cabello@fmf.uni-lj.si (S. Cabello), david.gajser@fmf.uni-lj.si (D. Gajser).

http://dx.doi.org/10.1016/j.dam.2015.03.004
0166-218X/

Algorithm 1: INDEPENDENT(h, G, ε)

Input: An integer $h > 0$, a K_h-minor-free graph $G = (V, E)$, and a parameter $\varepsilon \in (0, 1)$
Output: An independent set U of G
$r = C_h/\varepsilon^2$, where C_h is an appropriate constant depending on h
$U = \emptyset$
while $\exists U_1 \subseteq U, V_1 \subseteq V \setminus U$ *with* $|U_1| < |V_1| \le r$ *and* $(U \setminus U_1) \cup V_1$ *is an independent set* **do**
    $U = (U \setminus U_1) \cup V_1$
return U

Fig. 1. PTAS for MAXIMUM INDEPENDENT SET for K_h-minor-free graphs.

VERTEX COVER and MINIMUM DOMINATING SET. Baker [3] developed a technique to obtain PTAS for planar graphs using more elementary tools. In fact, much of the work for minor-free families is a vast, complex generalization of the approach by Baker.

To show how simple is our approach, look at the algorithm INDEPENDENT(h, G, ε) for MAXIMUM INDEPENDENT SET shown in Fig. 1. The algorithms for MINIMUM VERTEX COVER and MINIMUM DOMINATING SET are similar and provided in Section 3. In the algorithms we use a constant C_h that depends only on the size of the forbidden minor. Its actual value is in $\Theta(h^3)$, as we shall see.

We see that, for any fixed h, the algorithm is a very simple local optimization that returns an independent set that is $O(\varepsilon^{-2})$-locally optimal, in the sense that it cannot be made larger by substituting any $O(\varepsilon^{-2})$ of its vertices. The algorithm runs in time $n^{O(\varepsilon^{-2})}$, for any fixed h.

The main idea in the proof of the correctness of our algorithm is dividing the input graph into not-too-many pieces with $O(\varepsilon^{-2})$ vertices and small boundary, as defined in Section 2. For this we use the existence of separators [2] in the same way as Frederickson [8] did for planar graphs. A similar division has been used in other works; see for example [16]. The division is useful for the following fact: changing the solution U within one of the pieces cannot result in a better solution because U is $O(\varepsilon^{-2})$-locally optimal. Using this, we can infer (after some work) that, if G is K_h-minor-free, then

$$opt - |U| \le \varepsilon \cdot opt.$$

For MINIMUM VERTEX COVER and MINIMUM DOMINATING SET one has to make the additional twist of considering a division in a graph that represents the locally optimal solution and the optimal solution.

It is important to note that the analysis of the algorithm uses separators but the algorithm does not use them. Thus, all the difficulty is in the proof that the algorithm is a PTAS, not in the description of the algorithm. In any case, our proofs only rely on the existence of separators and is dramatically simpler than previous proofs of existence of PTAS's for MINIMUM VERTEX COVER and MINIMUM DOMINATING SET. In particular, we do not need any of the tools developed for the Graph Minor Theorem. A drawback of our method is that the running time is $n^{O(\varepsilon^{-2})}$, while previous, more complicated methods require $O(f(\varepsilon)n^c)$, for some constant $c > 0$ and function f. Another drawback of our method is that it works only in unweighted problems.

The idea of using separators to show that a local-optimization algorithm is a PTAS was presented by Chan and Har-Peled [5] and independently by Mustafa and Ray [14]. Local search was also used earlier to obtain constant-factor approximations by Agarwal and Mustafa [1]. The technique has been used recently to provide PTAS's for some geometric problems; see for example [4,6,9,13]. However, the use for minor-free families of graphs has passed unnoticed.

2. Dividing minor-free graphs

In this section we present a way of dividing a graph into subgraphs with special properties. We will not use this division in our algorithms, but it will be the main tool for their analysis.

Let G be a graph and let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a collection of subsets of vertices of G. We define the *boundary* of a piece $S_i \in \mathcal{S}$ (with respect to $\mathcal{S}$), denoted by ∂S_i, as those vertices of S_i that appear in some other piece $S_j \in \mathcal{S}, j \ne i$. Thus $\partial S_i = S_i \cap \left(\bigcup_{j \ne i} S_j\right)$. We define the *interior* of S as $\mathrm{int}(S_i) = S_i \setminus \partial S_i$.

A *division* of a graph G is a collection $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ of subsets of vertices of G satisfying the following two properties:

- $G = \bigcup_i G[S_i]$, that is, each edge and each vertex of G appears in some induced subgraph $G[S_i]$, and
- for each $S_i \in \mathcal{S}$ and $v \in \mathrm{int}(S_i)$, all neighbors of v are in S_i.

We refer to each subset $S_i \in \mathcal{S}$ as a *piece* of the division. (It may be useful to visualize a piece as the induced subgraph $G[S_i]$, since we actually use S_i as a proxy to $G[S_i]$.)

We want to find a division of a K_h-minor-free graph G where, for some parameter r that we can choose, each piece has roughly r vertices and all pieces together have roughly $|V(G)|/\sqrt{r}$ boundary vertices, counted with multiplicity. For technical