Discrete Applied Mathematics 187 (2015) 1–11

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The proportional partitional Shapley value

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ARTICLE INFO

Article history:
Received 16 December 2013
Received in revised form 20 October 2014
Accepted 20 January 2015
Available online 7 March 2015

Keywords:
Game theory
(TU) cooperative game
Shapley value
Coalition structure
Aumann–Drèze value

ABSTRACT

A new coalitional value is proposed under the hypothesis of isolated unions. The main difference between this value and the Aumann–Drèze value is that the allocations within each union are not given by the Shapley value of the restricted game but proportionally to the Shapley value of the original game. Axiomatic characterizations of the new value, examples illustrating its application and a comparative discussion are provided.

1. Introduction

The cooperative game theory deals with situations where a group of agents (players) want to share the benefits derived from their cooperation. It offers mathematical tools to propose, according to different criteria, allocation vectors that could be acceptable for the agents. This theory has given rise to relevant applications in many fields (see e.g. [5]).

Among those mathematical tools there are the so-called *values*. A value proposes for every cooperative game an allocation vector that represents a fair compromise for the players. Probably, the most important value is the Shapley value [12], denoted here by Φ. Moretti and Patrone [8] is a survey that shows the impact of the Shapley value in several scientific disciplines.

The notion of cooperative game with a *coalition structure* (a partition of the set N of players into *unions*) was considered in [2], and a modification of the Shapley value was proposed. Later on, other *coalitional values* (i.e. values for cooperative games with a coalition structure) have been introduced and analysed in the game theoretical literature. The two most cited coalitional values are the Aumann–Drèze value, denoted here by α, and the Owen value [11], denoted here by Ω. They are based on two different interpretations of the coalition structure that give rise to two different approaches when defining coalitional values:

1. Aumann and Drèze consider that, once a partition $\{P_1, \ldots, P_m\}$ of N has been formed, m independent cooperative situations arise (*isolated unions*), so their value allocates the benefits generated by each P_k to its members by applying the Shapley value to the restricted game.

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http://dx.doi.org/10.1016/j.dam.2015.01.031

2. Instead, Owen considers the partition rather as a way to influence the negotiation among the agents (*bargaining unions*), so his value allocates the benefits generated by N by applying the Shapley value twice: first, to sharing the total utility among the unions and, then, to sharing among the members of each union the payoff obtained in the first step.[1]

Example 1 (*A Glove Game*)**.** To illustrate both approaches, let us consider an elementary glove game with three players where player 1 has two right gloves and players 2 and 3 have one left glove each. Only each left-and-right pair of gloves has a worth of 1; otherwise, the worth is 0. The cooperative game v associated to this situation is given by

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$
$$v(\{1,2\}) = v(\{1,3\}) = 1, \qquad v(N) = 2.$$

Consider now that partition $P = \{\{1,2\},\{3\}\}$ forms. The Aumann–Drèze value yields the allocation $\alpha(v, P) = (1/2, 1/2, 0)$. Indeed, once P is formed, this value merely takes into account that players 1 and 2 are symmetric (in P_1) and must share 1 unit, whereas player 3 is a null player (in P_3). Instead, the Owen value yields the allocation $\Omega(v, P) = (1, 1/2, 1/2)$. It first allocates to the unions 3/2 and 1/2, respectively, and assigns then 1 to player 1, 1/2 to player 2, and 1/2 to player 3. Note that the shared worth is different.

In this paper we adopt approach 1, thus leaving aside the Owen value definitely, and introduce a new coalitional value, called the proportional partitional Shapley value and denoted as π, as an alternative to the Aumann–Drèze value. Hence we assume that, once a partition forms, a new cooperative situation arises in each union independently of the remaining ones. However, we wish to take into account in some manner the outside options of the players, reflected by the Shapley value of the original game. More precisely, given a cooperative game v in N with a coalition structure $P = \{P_1, \ldots, P_m\}$, our value divides each worth $v(P_k)$ among the players in P_k proportionally to the Shapley value of these players in game v.

Thus, in Example 1 we obtain the allocation $\pi(v, P) = (2/3, 1/3, 0)$ since the Shapley value is $\Phi(v) = (1, 1/2, 1/2)$. It reflects that player 1 is in a better position than player 2 because he might join player 3 if $\{1, 2\}$ collapses. We will restrict the domain of our value to the class of monotonic games in order to avoid some problems that often arise when using proportionality.

Example 2 (*A Second Glove Game*)**.** [2] Let $N = \{r, r, \ell, \ell, \ell, \ell\}$ be, informally, the set of players, each one with a glove: r means righty, ℓ means lefty. Only each left-and-right pair of gloves has a worth of one. The glove game v describing this is a linear combination of 45 unanimity games that we omit. The Shapley value is

$$\Phi(v) = \frac{1}{15}(11, 11, 2, 2, 2, 2)$$

and, for any partition $P = \{A, \ldots\}$, where $A = \{r, r, \ell\}$, the Aumann–Drèze and proportional partitional Shapley values respectively yield

$$\alpha(v, P) = \frac{1}{6}(1, 1, 4, 0, 0, 0) \quad \text{and} \quad \pi(v, P) = \frac{1}{24}(11, 11, 2, 0, 0, 0).$$

These allocations do not depend on the way the remaining three players ℓ are arranged (a general property of α and π). Instead, for the Owen value, this greatly matters. There are three possibilities:

$$P^1 = \{A, \{\ell\}, \{\ell\}, \{\ell\}\}, \qquad P^2 = \{A, B, \{\ell\}\} \quad \text{and} \quad P^3 = \{A, C\},$$

where $B = \{\ell, \ell\}$ and $C = \{\ell, \ell, \ell\}$. Thus for the Owen value we obtain

$$\Omega(v, P^1) = \frac{1}{12}(9, 9, 3, 1, 1, 1), \qquad \Omega(v, P^2) = \frac{1}{36}(25, 25, 10, 3, 3, 6), \quad \text{and}$$
$$\Omega(v, P^3) = \frac{1}{12}(7, 7, 4, 2, 2, 2).$$

This example is interesting. First, because it shows a difference between the Aumann–Drèze value and the proportional partitional Shapley value: the former is concerned with the possibilities existing in $A = \{r, r, \ell\}$ only, and hence it gives the bulk of the payoff to player ℓ; instead, the latter recalls the strategic strength in the original game, thus avoiding a striking change in the payout ratios that would not satisfy the righties. Second, it shows the main difference between the Aumann–Drèze and proportional partitional Shapley values and the Owen value. The former two satisfy local efficiency, whereas the latter satisfies efficiency, as the Shapley value does.

[1] The first sharing takes place in the *quotient game*, played by unions; the second sharing applies to games defined in each P_k that we will not describe. We refer the reader to [11].

[2] We are grateful to a reviewer for suggesting this numerical example.