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Set-reconstructibility of Post classes

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ABSTRACT

The clones of Boolean functions are classified in regard to set-reconstructibility via a strong dichotomy result: the clones containing only affine functions, conjunctions, disjunctions or constant functions are set-reconstructible, whereas the remaining clones are not weakly reconstructible.

1. Introduction

Reconstruction problems have been considered in various fields of mathematics and theoretical computer science, and they share the same meta-formulation: *given a family of "objects" and a systematic way of forming some sort of "derived objects", is an object uniquely determined (up to a sort of equivalence) by the collection of its derived objects?* It is possible that the same derived object arises from a given object in many different ways, and we usually keep track of the number of times each derived object arises; in other words, "collection" means the **multiset** of derived objects. On the other hand, if we ignore the numbers of occurrences of the derived objects, i.e., we take "collection" to mean the **set** of derived objects, then we are dealing with what is referred to as a set-reconstruction problem.

Several instances of this general formulation have become celebrated conjectures that have attracted a great deal of attention within the scientific community. Among these, the graph reconstruction conjecture (in both variants of vertex- or edge-deletion) [5,15] remains one of the most challenging that has survived as an open problem for many decades. Nonetheless, it has been shown to hold for numerous classes of graphs such as trees, and regular graphs. In fact, Bollobás [1] showed that the probability of finding a non-reconstructible graph tends to 0 as the number of vertices tends to infinity. In some other noteworthy instances, e.g., for directed graphs and hypergraphs, reconstructibility has been shown not to hold in general; see, e.g., [6,7,14]. Analogous reconstruction problems have been formulated for many other kinds of mathematical objects, such as relations (see Fraïssé [4]), ordered sets (see the survey by Rampon [13]), matrices (see Manvel and Stockmeyer [10]), integer partitions (see Monks [11]), and multisets (see Lehtonen [8]).

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Table 1
The four simple graphs on three vertices and their decks and set-decks.

G	K_3	P_3	$\overline{P_3}$	$\overline{K_3}$
deck G	K_2 K_2 K_2	K_2 K_2 $\overline{K_2}$	K_2 $\overline{K_2}$ $\overline{K_2}$	$\overline{K_2}$ $\overline{K_2}$ $\overline{K_2}$
set-deck G	K_2	K_2 $\overline{K_2}$	K_2 $\overline{K_2}$	$\overline{K_2}$

Example 1.1. In order to illustrate these notions, let us consider a simple case of the reconstruction problem for simple graphs and one-vertex-deleted subgraphs. There are four non-isomorphic simple graphs on three vertices, as shown on the first row of Table 1. From each one of these graphs G, we form three subgraphs by deleting one of the three vertices and the edges incident to it. If we keep track of the number of times each graph on two vertices arises from G in this way, then we obtain the deck of G, as shown on the second row of Table 1. As can be easily seen from the table, the four simple graphs have distinct decks. Therefore, they are all uniquely determined by their decks. In other words, the simple graphs on four vertices are reconstructible. On the other hand, if we ignore the multiplicities of the subgraphs, then we obtain the set-decks of the graphs, as shown on the third row of Table 1. Now it happens that the graphs P_3 and $\overline{P_3}$ have identical set-decks, so they are not set-reconstructible. Nevertheless, the graphs K_3 and $\overline{K_3}$ are set-reconstructible.

In this paper we consider a reconstruction problem for functions of several arguments, taking the identification of a pair of arguments as the way of forming derived objects: is a function $f : A^n \to B$ determined (up to equivalence) by its identification minors?

Lehtonen [8,9] answers this question positively for certain function classes such as those of symmetric functions or affine functions. Recently, we showed [3] that the class of order-preserving functions is not reconstructible, even if restricted to lattice polynomial functions. In the case of Boolean functions, the latter result refines into a classification of Post classes (clones of Boolean functions): the only reconstructible Post classes are the ones containing only affine functions, conjunctions, disjunctions or constant functions. The remaining Post classes are not weakly reconstructible.

The purpose of this paper is to make this dichotomy of Post classes even more contrasting: the reconstructible Post classes are actually set-reconstructible. This shows that reconstructibility is the same as set-reconstructibility in this setting.

The paper is organized as follows. In Section 2 we recall the basic notions, state preliminary results and formulate the reconstruction problem for functions of several arguments and identification minors. We focus on the (set)-reconstructibility of clones of operations in Section 3, where we provide a dichotomy theorem dealing with the set-reconstructibility of Post classes.

2. Preliminaries

2.1. General

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. Throughout this paper, k, ℓ, m and n stand for positive integers, and A and B stand for arbitrary finite sets with at least two elements. The set $\{1, \ldots, n\}$ is denoted by $[n]$. The set of all 2-element subsets of a set A is denoted by $\binom{A}{2}$. Tuples are denoted by bold-face letters and components of a tuple are denoted by the corresponding italic letters with subscripts, e.g., $\mathbf{a} = (a_1, \ldots, a_n)$.

Let $\mathbf{a} \in A^n$, and let $\sigma : [m] \to [n]$. We will write $\mathbf{a}\sigma$ to denote the m-tuple $(a_{\sigma(1)}, \ldots, a_{\sigma(m)})$. Since the n-tuple $\mathbf{a}$ can be formally seen as the map $\mathbf{a} : [n] \to A$, $i \mapsto a_i$, the m-tuple $\mathbf{a}\sigma$ is just the composite map $\mathbf{a} \circ \sigma : [m] \to A$.

A *finite multiset* M on a set S is a couple $(S, \mathbf{1}_M)$, where $\mathbf{1}_M : S \to \mathbb{N}$ is a map, called a *multiplicity function*, such that the set $\{x \in S : \mathbf{1}_M(x) \neq 0\}$ is finite. Then the sum $\sum_{x \in S} \mathbf{1}_M(x)$ is a well-defined natural number, and it is called the *cardinality* of M. For each $x \in S$, the number $\mathbf{1}_M(x)$ is called the *multiplicity* of x in M. If $(a_i)_{i \in I}$ is a finite indexed family of elements of S, then we will write $\langle a_i : i \in I \rangle$ to denote the multiset in which the multiplicity of each $x \in S$ equals $|\{i \in I : a_i = x\}|$.