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Bounds on the disparity and separation of tournament solutions

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ABSTRACT

A tournament solution is a function that maps a tournament, i.e., a directed graph representing an asymmetric and connex relation on a finite set of alternatives, to a non-empty subset of the alternatives. Tournament solutions play an important role in social choice theory, where the binary relation is typically defined via pairwise majority voting. If the number of alternatives is sufficiently small, different tournament solutions may return overlapping or even identical choice sets. For two given tournament solutions, we define the *disparity index* as the order of the smallest tournament for which the solutions differ and the *separation index* as the order of the smallest tournament for which the corresponding choice sets are disjoint. Isolated bounds on both values for selected tournament solutions are known from the literature. In this paper, we address these questions systematically using an exhaustive computer analysis. Among other results, we provide the first tournament in which the bipartisan set and the Banks set are not contained in each other.

1. Introduction

An important area in the mathematical social sciences concerns solution concepts that identify desirable sets of alternatives based on the preferences of multiple agents. Many of these concepts are defined in terms of a so-called dominance relation, where one alternative dominates another if a strict majority of the agents prefer the former to the latter. This relation can be nicely represented as an oriented graph whose vertices are the alternatives and there is a directed edge from a to b if and only if a dominates b. Whenever there is an odd number of agents with linear preferences, the dominance relation is asymmetric and connex, i.e. there is exactly one directed edge between any pair of distinct vertices, and the graph thus constitutes a tournament. A tournament solution is a function that maps a tournament to a non-empty subset of its vertices or alternatives. Application areas of tournament solutions include voting [39,35], multi-criteria decision analysis [2,4], zero-sum games [27,34,23], and coalitional games [8].

A wide variety of tournament solutions have been proposed in the literature. Even though many of them are based on vastly different ideas, they share some similarities. For instance, all tournament solutions considered in this paper uniquely select the Condorcet winner, i.e. an alternative that dominates every other alternative, whenever such an alternative exists. Moreover, some tournament solutions return completely identical or at least overlapping choice sets if the number of alternatives is sufficiently small. In this paper, we aim at formalizing and systematically investigating the similarity of any given pair of tournament solutions by studying the minimal number of alternatives that are required for the disparity and the separation of the corresponding choice sets. To this end, we define the *disparity index* as the order of the smallest tournament

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for which the solutions differ and the *separation index* as the order of the smallest tournament for which the corresponding choice sets are disjoint.

Isolated bounds on both values for selected tournament solutions have been provided in previous work. In particular, the construction of tournaments for which certain tournament solutions return disjoint choice sets has been addressed by several researchers. For example, the first tournament proposed in the literature for which the Banks set and the Slater set are disjoint consists of 75 alternatives [33].[1] Later, this bound on the separation index was improved to 16 alternatives by Charon et al. [19] and, more recently, to 14 alternatives by Östergård and Vaskelainen [40]. Östergård and Vaskelainen have also provided a lower bound of 11 by means of an exhaustive computer analysis. In other work, Hudry [31] has shown that the separation index for the Banks set and the Copeland set is 13. Dutta [25] provided a tournament of order 8 in which the Banks set and the tournament equilibrium set are both strictly contained in the minimal covering set. Among other facts, our study has shown that Dutta's example is minimal.

Perhaps the most interesting open problem regarding the relationships between tournament solutions concerns the bipartisan set and the Banks set. In all examples studied so far, either the Banks set is contained in the bipartisan set or the Banks set is contained in the bipartisan set (see, e.g. [35]). In particular, it is unknown whether these tournament solutions always intersect. In this paper, we provide the first tournament in which the bipartisan set and the Banks set are *not* contained in each other. This tournament is of order 8. The minimal covering set (a superset of the bipartisan set) has been shown to always intersect with the Banks set. We show that the smallest tournament in which neither choice set is contained in the other is of order 10. Our findings are summarized in Sections 4 and 5.

2. Preliminaries

A (finite) *tournament* T is a pair $(A, \succ)$, where A is a set of alternatives and $\succ$ is an asymmetric and connex (but not necessarily transitive) binary relation on A, usually referred to as the *dominance relation*. Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. Moreover, for a subset of alternatives $B \subseteq A$, we will sometimes consider the restriction of the dominance relation $\succ_B = \succ \cap (B \times B)$.

For a tournament $(A, \succ)$ and an alternative $a \in A$, we denote by $D(a)$ the *dominion* (or out-neighborhood) of a, i.e.

$$D(a) = \{b \in A \mid a \succ b\},$$

and by $\overline{D}(a)$ the set of *dominators* (or in-neighborhood) of a, i.e.

$$\overline{D}(a) = \{b \in A \mid b \succ a\}.$$

The *order* $|T|$ of a tournament $T = (A, \succ)$ refers to the cardinality of A, and $\mathcal{T}_n$ denotes the set of all tournaments of order n or less. The set of all linear orders on some set A is denoted by $\mathcal{L}(A)$ and the maximal element of A according to a linear order $L \in \mathcal{L}(A)$ is denoted by $\max(L)$.

The elements of the *adjacency matrix* $M(T) = (m_{ab})_{a,b\in A}$ of a tournament T are 1 whenever $a \succ b$ and 0 otherwise. The *skew-adjacency matrix* $G(T)$ of the corresponding tournament graph is skew-symmetric and defined as the difference of the adjacency matrix and its transpose, i.e. $G(T) = M(T) - M(T)^t$.

A *tournament solution* is a function that maps a tournament to a nonempty subset of its alternatives. For two tournament solutions S_1 and S_2, we define the *disparity index* $d(S_1, S_2)$ as the order of the smallest tournament T for which S_1 and S_2 differ, i.e.

$$d(S_1, S_2) = \min\{n \in \mathbb{N} \mid \exists T \in \mathcal{T}_n \text{ such that } S_1(T) \neq S_2(T)\}.$$

Similarly, we define the *separation index* $s(S_1, S_2)$ as the order of the smallest tournament T for which the two respective choice sets are disjoint. Formally,

$$s(S_1, S_2) = \min\{n \in \mathbb{N} \mid \exists T \in \mathcal{T}_n \text{ such that } S_1(T) \cap S_2(T) = \emptyset\}.$$

Obviously, $d(S_1, S_2) \leq s(S_1, S_2)$ for all tournament solutions S_1 and S_2.

We now define the tournament solutions considered in this paper and address the question of how to compute them. For an overview and more details on most concepts, we refer to Laslier [35] and Brandt et al. [13]. Computational issues are discussed by Brandt et al. [13], Hudry [32], and Brandt [5].

Copeland set. The *Copeland set* $CO(T)$ [21] of a tournament T consists of all alternatives whose dominion is of maximum size, i.e.

$$CO(T) = \arg\max_{a \in A} |D(a)|.$$

$|D(a)|$ is also called the *Copeland score* of a. This set can be easily computed in time $O(|T|^2)$ by determining all out-degrees and choosing the alternatives with maximum out-degree.

[1] Laffond and Laslier [33] presented a similar tournament on 139 alternatives in which the Banks set, the Slater, and the Copeland set are all disjoint from each other.