Discrete Applied Mathematics 187 (2015) 91–95

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A characterization of the non-trivial diameter two graphs of minimum size

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ARTICLE INFO

Article history:
Received 14 May 2014
Received in revised form 16 January 2015
Accepted 9 February 2015
Available online 16 March 2015

Keywords:
Diameter two graphs
Bounds
Size

ABSTRACT

Let G be a graph with diameter two, order n and size m, such that no vertex is adjacent to every other vertex. A classical result due to Erdős and Rényi (1962) states that $m \geq 2n - 5$. We characterize the graphs that achieve equality in the Erdős–Rényi bound.

1. Introduction

Distance and diameter are fundamental concepts in graph theory. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest u–v path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by $\text{diam}(G)$. We say that G is a *diameter-k graph* if $\text{diam}(G) = k$. A vertex of G adjacent to every other vertex in G we call a *dominating vertex* of G. We begin with the following trivial observation.

Observation 1. *If G is a diameter-2 graph of order n and size m, then $m \geq n - 1$ with equality if and only if G is the star graph $K_{1,n-1}$.*

If we forbid graphs with a dominating vertex, the situation becomes more interesting and significantly more edges are required. Erdős and Rényi [4] proved the following classical result on the minimum size of a diameter-2 graph with no dominating vertex.

Erdős–Rényi Theorem ([4]). *If G is a diameter-2 graph of order n and size m with no dominating vertex, then $m \geq 2n - 5$.*

In this paper we characterize the graphs that achieve equality in the lower bound established in the Erdős–Rényi Theorem. We note that a generalized version of the Erdős–Rényi Theorem also appears in Bollobás' book Extremal Graph Theory [1] without the characterization.

1.1. Notation

For notation and graph theory terminology not defined herein, we refer the reader to [2]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. Let v be a vertex in $V(G)$ and let X

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http://dx.doi.org/10.1016/j.dam.2015.02.009

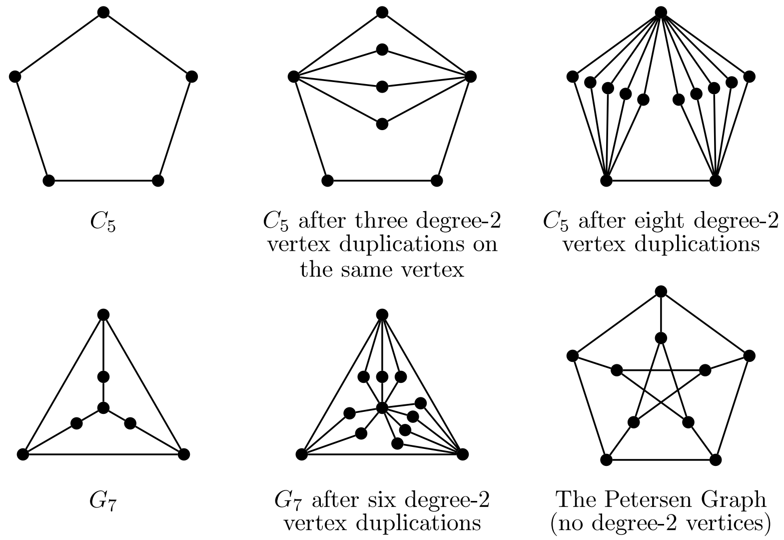

Fig. 1. Some examples of graphs in the family $\mathcal{G}$.

and Y be subsets of $V(G)$. We denote the *degree* of v in G by $d_G(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The *open neighborhood* of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. We let $N_G^X(v)$ denote the set of neighbors of v in G that belong to the set X and $d_G^X(v)$ denote the number of vertices in X adjacent to v in G. The *open neighborhood* of X is the set $N_G(X) = \bigcup_{u \in X} N(u)$, and its *closed neighborhood* is the set $N_G[X] = N(X) \cup X$. We say that X *dominates* G if $N[X] = V(G)$. A *dominating edge* of G is an edge of G such that every vertex is adjacent to at least one of its ends.

The subgraph induced by X is denoted by $G[X]$. If $X \neq V(G)$, then we denote the graph obtained from G by deleting all vertices in X as well as any incident edges by $G - X$. If X consists of a single vertex u, we simply denote $G - X$ by $G - u$. We denote the set of edges that join a vertex of X and a vertex of Y in G by $G[X, Y]$, or simply $[X, Y]$ if the graph G is clear from context.

By *identifying* two distinct vertices u and v in G we shall mean removing u and v from G and replacing them with a new vertex joined to every vertex in $N_G(\{u, v\}) \setminus \{u, v\}$. By *duplicating* the vertex v in G we shall mean adding a new vertex to the graph G and joining it to every vertex in $N_G(v)$. If G contains a vertex of degree 2, then we define *degree-*2 *vertex duplication on* G as the operation that produces a new graph from G by duplicating any vertex of degree 2. By *destructively duplicating* the vertex v in G we shall mean duplicating the vertex with a new vertex v' and then deleting one or more of the edges incident with vertices in $\{v, v'\}$.

2. The family $\mathcal{G}$ of diameter-2 graphs

In this section we define two special families of graphs. Let G_7 (depicted in Fig. 1) be the graph obtained from a 3-cycle by adding a pendant edge to each vertex of the cycle and then adding a new vertex and joining it to the three degree 1 vertices. Let $\mathcal{G}$ be the family of graphs that: (i) contains C_5, G_7 and the Petersen graph; and (ii) is closed under degree-2 vertex duplication. (See Fig. 1.)

3. Main results

Our aim in this paper is to characterize the graphs that achieve equality in the lower bound established in the Erdős–Rényi Theorem. We shall prove the following result, a proof of which is given in Section 5.

Theorem 1. *If G is a diameter-*2 *graph of order n and size m with no dominating vertex, then $m \geq 2n - 5$ with equality if and only if $G \in \mathcal{G}$.*

4. Preliminary results and observations

In this section, we present some preliminary results and observations that we will need when proving our main result. By construction of graphs in the family $\mathcal{G}$, we note that no graph in $\mathcal{G}$ has a dominating vertex. Consequently we have the following observation.

Observation 2. *If $G \in \mathcal{G}$ and u and v are two adjacent vertices in G, then the graph obtained from G by adding a new vertex, w, and adding the edges uw and vw has diameter* 3.