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Trees with large neighborhood total domination number

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ABSTRACT

In this paper, we continue the study of neighborhood total domination in graphs first studied by Arumugam and Sivagnanam (2011). A neighborhood total dominating set, abbreviated NTD-set, in a graph G is a dominating set S in G with the property that the subgraph induced by the open neighborhood of the set S has no isolated vertex. The neighborhood total domination number, denoted by $\gamma_{\rm nt}(G)$, is the minimum cardinality of a NTD-set of G. Every total dominating set is a NTD-set, implying that $\gamma(G) \le \gamma_{\rm nt}(G) \le \gamma_t(G)$, where $\gamma(G)$ and $\gamma_t(G)$ denote the domination and total domination numbers of G, respectively. Arumugam and Sivagnanam posed the problem of characterizing the connected graphs G of order $n \ge 3$ achieving the largest possible neighborhood total domination number, namely $\gamma_{\rm nt}(G) = \lceil n/2 \rceil$. A partial solution to this problem was presented by Henning and Rad (2013) who showed that 5-cycles and subdivided stars are the only such graphs achieving equality in the bound when n is odd. In this paper, we characterize the extremal trees achieving equality in the bound when n is even. As a consequence of this tree characterization, a characterization of the connected graphs achieving equality in the bound when n is even can be obtained noting that every spanning tree of such a graph belongs to our family of extremal trees.

1. Introduction

In this paper we continue the study of a parameter, called the neighborhood total domination number, that is squeezed between arguably the two most important domination parameters, namely the domination number and the total domination number. A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [3,4]. Total domination is now well studied in graph theory. For a recent book on the topic, see [8]. A survey of total domination in graphs can also be found in [5].

Arumugam and Sivagnanam [1] introduced and studied the concept of neighborhood total domination in graphs. A *neighbor* of a vertex v is a vertex different from v that is adjacent to v. The *neighborhood of a set S* is the set of all neighbors of vertices in S. A *neighborhood total dominating set*, abbreviated NTD-set, in a graph G is a dominating set S in G with the property that the subgraph induced by the open neighborhood of the set S has no isolated vertex. The *neighborhood total domination number* of G, denoted by $\gamma_{\rm nt}(G)$, is the minimum cardinality of a NTD-set of G. A NTD-set of G of cardinality $\gamma_{\rm nt}(G)$ is called a $\gamma_{\rm nt}(G)$-*set*.

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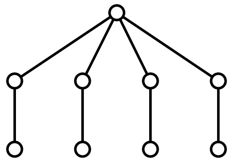

Fig. 1. A subdivided star.

Every TD-set is a NTD-set, while every NTD-set is a dominating set. Hence the neighborhood total domination number is bounded below by the domination number and above by the total domination number as first observed by Arumugam and Sivagnanam in [1].

Observation 1 ([1,6]). *If G is a graph with no isolated vertex, then* $\gamma(G) \le \gamma_{\rm nt}(G) \le \gamma_t(G)$.

1.1. Terminology and notation

For notation and graph theory terminology not defined herein, we refer the reader to [3]. Let G be a graph with vertex set $V(G)$ of order $n = |V(G)|$ and edge set $E(G)$ of size $m = |E(G)|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote the set of leaves in G by $L(G)$, and the set of support vertices by $S(G)$. A support vertex adjacent to two or more leaves is a *strong support vertex*. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A 2-*packing* in G is a set of vertices that are pairwise at distance at least 3 apart in G.

A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. A *star* on $n \ge 2$ vertices is a tree with a vertex of degree $n-1$ and is denoted by $K_{1,n-1}$. A *double star* is a tree containing exactly two vertices that are not leaves (which are necessarily adjacent). A *subdivided star* is a graph obtained from a star on at least two vertices by subdividing each edge exactly once. The subdivided star obtained from a star $K_{1,4}$, for example, is shown in Fig. 1. We note that the smallest two subdivided stars are the paths P_3 and P_5. Let $\mathcal{F}$ be the family of all subdivided stars. Let $F \in \mathcal{F}$. If $F = P_3$, we select a leaf of F and call it the *link vertex* of F, while if $F \ne P_3$, the *link vertex* of F is the central vertex of F.

The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. As observed in [6] a NTD-set in G is a set S of vertices such that $N[S] = V$ and $G[N(S)]$ contains no isolated vertex.

A *rooted tree* distinguishes one vertex r called the *root*. For each vertex $v \ne r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex u such that the unique (r, u)-path contains v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

2. Known results

The following upper bound on the neighborhood total domination number of a connected graph in terms of its order is established in [6].

Theorem 2 ([6]). *If G is a connected graph of order* $n \ge 3$, *then* $\gamma_{\rm nt}(G) \le (n+1)/2$.

In this paper we consider the following problem posed by Arumugam and Sivagnanam [1] to characterize the connected graphs of largest possible neighborhood total domination number.

Problem 1 ([1]). Characterize the connected graphs G of order n for which $\gamma_{\rm nt}(G) = \lceil n/2 \rceil$.

A partial solution to this problem was presented by Henning and Rad [6] who provided the following characterization in the case when n is odd.

Theorem 3 ([6]). *Let* $G \ne C_5$ *be a connected graph of order* $n \ge 3$. *If* $\gamma_{\rm nt}(G) = (n+1)/2$, *then* $G \in \mathcal{F}$.

As first observed in [6], a characterization in the case when n is even and the minimum degree is at least 2 follows readily from a result on the restrained domination number of a graph due to Domke, Hattingh, Henning and Markus [2]. Let $B_1, B_2, \ldots, B_5$ be the five graphs shown in Fig. 2.

Theorem 4 ([6]). *Let* $G \ne C_5$ *be a connected graph of order* $n \ge 4$ *with* $\delta(G) \ge 2$. *If* $\gamma_{\rm nt}(G) = n/2$, *then* $G \in \{B_1, B_2, B_3, B_4, B_5\}$.

3. The family $\mathcal{T}$ of trees

In this section we define a family of trees $\mathcal{T}$ as follows. Let T_0 be an arbitrary tree. Let T_1 be the tree obtained from T_0 by the following operation: for each vertex $x \in V(T_0)$, either add a new vertex and an edge joining it to x or add a new path P_3