Discrete Applied Mathematics 187 (2015) 103–110

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The difference between remoteness and radius of a graph

Hongbo Hua [a,b,*], Yaojun Chen [b], Kinkar Ch. Das [c]

[a] Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an, Jiangsu 223003, PR China

[b] Department of Mathematics, Nanjing University, Nanjing 210093, PR China

[c] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

ARTICLE INFO

Article history:
Received 17 April 2014
Received in revised form 2 November 2014
Accepted 4 February 2015
Available online 6 March 2015

Keywords:
Remoteness
Radius
Diameter
Low bound
Conjecture

ABSTRACT

Let G be a connected graph of order $n \geq 3$. The remoteness $\rho = \rho(G)$ is the maximum, over all vertices, of the average distance from a vertex to all others. The radius $r = r(G)$ is the minimum, over all vertices, of the eccentricity of a vertex. Aouchiche and Hansen (2011) conjectured that $\rho - r \geq \frac{3-n}{4}$ if n is odd and $\rho - r \geq \frac{2n-n^2}{4(n-1)}$ if n is even. In this paper, we confirm this conjecture. In addition, we completely characterize extremal graphs attaining the lower bound.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. For a vertex v in $V(G)$, we let $N_G(v)$ be the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$ be the degree of v. Let G' be a subgraph of G and $v \in V(G')$. Then $d_{G'}(v)$ is equal to the number of edges incident to v in G'. Let P_n and C_n be the path and cycle on n vertices, respectively. For vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is defined as the length of the shortest path connecting u and v in G and $D_G(v)$ is equal to the sum of distances between v and all other vertices in G, that is, $D_G(v) = \sum_{u \in V(G)} d_G(v, u)$. The *eccentricity* $\varepsilon_G(v)$ of a vertex v is the maximum distance from v to any other vertex in G, i.e., $\varepsilon_G(v) = \max_{u \in V(G)} d_G(v, u)$. The *radius* of a connected graph G is defined by $r = r(G) = \min_{v \in V(G)} \varepsilon_G(v)$, and the *diameter* of a connected graph G is defined by $D = D(G) = \max_{v \in V(G)} \varepsilon_G(v)$. The *proximity* $\pi = \pi(G)$ of a connected graph G is the minimum, over all vertices, of the average distance from a vertex to all others. The *remoteness* $\rho = \rho(G)$ of a connected graph G is the maximum of the average distance from a vertex to all others. That is,

$$\pi = \pi(G) = \min_{v \in V(G)} \frac{1}{n-1} D_G(v)$$

and

$$\rho = \rho(G) = \max_{v \in V(G)} \frac{1}{n-1} D_G(v).$$

The sum of distances from a vertex to all others is also known as its transmission. So π and ρ can also be seen as the minimum and maximum normalized (divided by the number of vertices implied in the sum) transmission in a graph. It

* Corresponding author at: Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an, Jiangsu 223003, PR China.
E-mail addresses: hongbo.hua@gmail.com (H. Hua), yaojunc@nju.edu.cn (Y. Chen), kinkardas2003@gmail.com (K.Ch. Das).

http://dx.doi.org/10.1016/j.dam.2015.02.007
0166-218X/

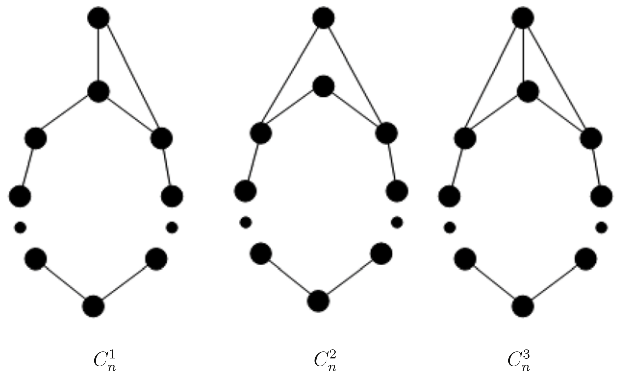

Fig. 1. Graphs C_n^1, C_n^2 and C_n^3 (n is odd).

is interesting to normalize the transmission in order to get invariants that fit in the same range as the radius, diameter, average distance and average eccentricity, and that preserve the properties of the transmission. The above two invariants were proposed in [2,3]. Nordhaus–Gaddum relations for both of them, i.e., lower and upper bounds on $\pi(G)+\pi(\overline{G})$, $\pi(G)\cdot\pi(\overline{G})$, $\rho(G)+\rho(\overline{G})$, $\rho(G)\cdot\rho(\overline{G})$, are given in [3].

In [4], Aouchiche and Hansen compared these two graph invariants π and ρ to the diameter, radius, average eccentricity, average distance, independence number and matching number. Most bounds so obtained were proved, but a few of them remained as conjectures. These proved results and conjectures stated in [4] were obtained with the use of AutoGraphiX, a conjecture-making system in graph theory [1,5,6]. Recently, some authors including two of the present authors solved several conjectures from AutoGraphiX (see [8,7,9–13]).

Among those results and conjectures mentioned in [4], there are two dealing with the difference between the remoteness ρ and radius r. Aouchiche and Hansen have found a sharp upper bound on the difference $\rho - r$. However, the lower bound on the difference $\rho - r$ was unsolved by them and remained as a conjecture (see [4]), which reads as follows.

Conjecture 1. *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. Then $\rho - r \geq \frac{3-n}{4}$ if n is odd and $\rho - r \geq \frac{2n-n^2}{4(n-1)}$ if n is even. The inequality is best possible as shown by the cycle C_n if n is even and by the graph composed by the cycle C_n together with two crossed edges on four successive vertices of the cycle.*

Recently, this conjecture has been proved to be correct for the case of trees by Sedlar [15]:

Theorem 1 (*Sedlar [15]*)**.** *Let T be a tree on $n \geq 3$ vertices with remoteness ρ and radius r. If n is odd, then $\rho - r \geq \frac{1}{2}$ with equality if and only if $T \cong P_n$, and if n is even, then $\rho - r \geq \frac{n}{2(n-1)}$ with equality if and only if T is a tree obtained by attaching a pendent edge to a central vertex of the path P_{n-1}.*

This conjecture is still open so far. In this paper, we confirm that Conjecture 1 holds for all connected graphs of order $n \geq 3$. Moreover, we characterize all extremal graphs attaining the lower bound. Our main result is the following.

Theorem 2. *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. If n is odd, then $\rho - r \geq \frac{3-n}{4}$ with equality if and only if $G \cong C_n$ or C_n^i, $i = 1, 2, 3$ (see Fig. 1), and if n is even, then $\rho - r \geq \frac{2n-n^2}{4(n-1)}$ with equality if and only if $G \cong C_n$.*

Remark 1. Denote by C_n^3, a graph of odd order n, obtained from the cycle C_n together with two crossed edges on four successive vertices of the cycle. One can see that C_n^3 is just an extremal graph mentioned in Conjecture 1 for the case of odd n. Evidently, $\rho(C_n^3) - r(C_n^3) = \frac{3-n}{4}$.

Denote by C_n^k ($k = 1, 2$) (see Fig. 1.) graphs of odd order n, obtained from the cycle C_{n-1} by adding two additional edges between an isolated vertex u and each of two vertices v and w on the cycle C_{n-1} such that $d_{C_{n-1}}(v, w) = k$. It is easy to check that

$$\rho(C_n^k) - r(C_n^k) = \frac{3-n}{4}, \quad k = 1, 2.$$

Thus, each C_n^k is also a graph attaining the lower bound on $\rho - r$ of Conjecture 1 for the case of odd n, $k = 1, 2$.

In order to prove Theorem 2, we need additional terminologies and results.

Let u, v be two distinct nonadjacent vertices of a graph G and $S \subseteq V(G) - \{u, v\}$. If u and v belong to different components of $G - S$, then we say that S separates u and v.

The following is a well known result on connectivity of a graph due to Menger in 1927 [14].