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Resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs[☆]

Xiaogang Liu[a,b], Jiang Zhou[c,*], Changjiang Bu[c]

[a] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

[b] Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

[c] College of Science, Harbin Engineering University, Harbin 150001, PR China

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ABSTRACT

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *R-graph* of a graph G, denoted by $\mathcal{R}(G)$, is the graph obtained from G by adding a vertex v_e and then joining v_e to the end vertices of e for each $e \in E(G)$. Let G_1 and G_2 be two vertex disjoint graphs. The *R-vertex join* of G_1 and G_2, denoted by $G_1\langle \mathrm{v}\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$. The *R-edge join* of G_1 and G_2, denoted by $G_1\langle \mathrm{e}\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$, where $I(G_1)$ is the set of the added vertices of $\mathcal{R}(G_1)$. In this paper, we formulate the resistance distances and the Kirchhoff index of $G_1\langle \mathrm{v}\rangle G_2$ and $G_1\langle \mathrm{e}\rangle G_2$ respectively.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = \mathrm{diag}(d_1, d_2, \ldots, d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. Let A_G denote the *adjacency matrix* of G. The *Laplacian matrix* of G is defined as $L_G = D_G - A_G$. We use $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{|V(G)|}(G) = 0$ to denote the eigenvalues of L_G. If G is connected, then any principal submatrix of L_G is nonsingular.

Let G be a connected graph. The *resistance distance* between any two vertices u and v in G is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The *Kirchhoff index* of G is the sum of resistance distances between all pairs of vertices of G. As usual, let $\Omega_{uv}(G)$ denote the resistance distance between u and v in G and $\mathrm{Kf}(G)$ denote the Kirchhoff index of G. Up till now, many results on the resistance distance and the Kirchhoff index are obtained. See [2,3,5,7,11,12,15,17–26,28,29] and the references therein to know more.

The *R-graph* [9,16] of a graph G, denoted by $\mathcal{R}(G)$, is the graph obtained from G by adding a vertex v_e and then joining v_e to the end vertices of e for each $e \in E(G)$. We use $I(G)$ to denote the set of all added vertices of $\mathcal{R}(G)$. Based on R-graph, we define two new graph operations as follows.

Definition 1.1. Let G_1 and G_2 be two vertex disjoint graphs. The *R-vertex join* of G_1 and G_2, denoted by $G_1\langle \mathrm{v}\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$.

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* Corresponding author.
E-mail addresses: xiaogliu.yzhang@gmail.com (X. Liu), zhoujiang04113112@163.com (J. Zhou), buchangjiang@hrbeu.edu.cn (C. Bu).

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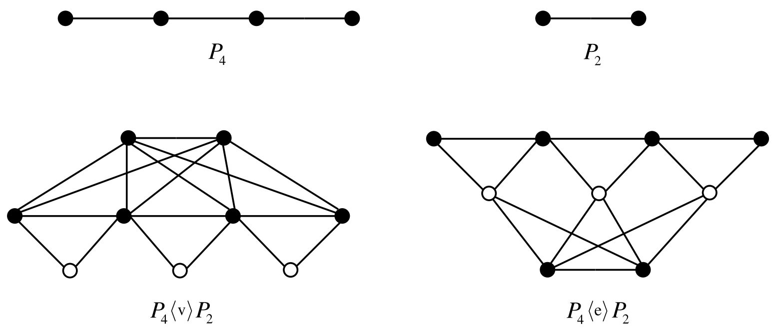

Fig. 1. An example of *R*-vertex join and *R*-edge join.

Definition 1.2. Let G_1 and G_2 be two vertex disjoint graphs. The *R-edge join* of G_1 and G_2, denoted by $G_1\langle e\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$.

Example 1.3. Let P_n denote a path of order n. Fig. 1 depicts the *R*-vertex join $P_4\langle v\rangle P_2$ and *R*-edge join $P_4\langle e\rangle P_2$, respectively.

In this paper, we formulate the resistance distances and the Kirchhoff index of $G_1\langle v\rangle G_2$ and $G_1\langle e\rangle G_2$, respectively.

2. Preliminaries

Let M be a square matrix. The $\{1\}$-*inverse* of M is a matrix X such that $MXM = M$. If M is singular, then M has infinitely many $\{1\}$-inverses [4]. The *group inverse* of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$, and $MX = XM$. It is known [4,6] that $M^{\#}$ exists if and only if $\mathrm{rank}(M) = \mathrm{rank}(M^2)$. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore–Penrose inverse of M if M is symmetric [6,14].

Let $M^{(1)}$ denote any $\{1\}$-inverse of a matrix M and let $(M)_{uv}$ denote the (u, v)-entry of M.

Lemma 2.1 ([1,6]). *Let G be a connected graph. Then*

$$\begin{aligned}\Omega_{uv}(G) &= \left(L_G^{(1)}\right)_{uu} + \left(L_G^{(1)}\right)_{vv} - \left(L_G^{(1)}\right)_{uv} - \left(L_G^{(1)}\right)_{vu}\\ &= \left(L_G^{\#}\right)_{uu} + \left(L_G^{\#}\right)_{vv} - 2\left(L_G^{\#}\right)_{uv}.\end{aligned}$$

For a vertex i of a graph G, let $\Gamma(i)$ denote the set of all neighbors of i in G.

Lemma 2.2 ([7,8]). *Let G be a connected graph. For any* $i, j \in V(G)$,

$$\Omega_{ij}(G) = d_i^{-1}\left(1 + \sum_{k\in\Gamma(i)}\Omega_{kj}(G) - d_i^{-1}\sum_{k,l\in\Gamma(i)}\Omega_{kl}(G)\right).$$

Let $\mathbf{1}_n$ denote the all-ones column vector of dimension n. We will often use $\mathbf{1}$ to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2.3 ([7]). *Let* $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3\end{pmatrix}$ *be the Laplacian matrix of a connected graph. If each column vector of* L_2^T *is* $-\mathbf{1}$ *or a zero vector, then* $N = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#}\end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of L, where* $S = L_3 - L_2^T L_1^{-1} L_2$.

Lemma 2.4 ([27]). *Let* $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$ *be a nonsingular matrix. If A is nonsingular, then*

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1}\end{pmatrix},$$

where $S = D - CA^{-1}B$.

Let I_n be the identity matrix of size n, and $J_{s\times t}$ the $s \times t$ matrix with all entries equal to one.

Lemma 2.5 ([7]). *Let G be a graph of order n. For any* $a > 0$, *we have*

$$\left(L_G + aI_n - \frac{a}{n}J_{n\times n}\right)^{\#} = (L_G + aI_n)^{-1} - \frac{1}{an}J_{n\times n}.$$