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The Minimum Flow Cost Hamiltonian Cycle Problem: A comparison of formulations

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ABSTRACT

We introduce the *Minimum Flow Cost Hamiltonian Cycle Problem* (FCHCP). Given a graph and positive flow between pairs of vertices, the FCHCP consists of finding a Hamiltonian cycle that minimizes the total flow cost between pairs of vertices through the shortest path on the cycle. We prove that the FCHCP is *NP*-hard and we study the polyhedral structure of its set of feasible solutions. In particular, we present five different mixed integer programming formulations which are theoretically and computationally compared. We also propose several families of valid inequalities for one of the formulations and perform some computational experiments to assess the performance of these inequalities.

1. Introduction

Network optimization is probably one of the most important and widely studied research areas in combinatorial optimization. *Network design problems* (NDP) consist of identifying an optimal subgraph of a graph and of routing flows between origin/destination (O/D) vertices, subject to some feasibility conditions [1,20]. These problems frequently arise in the design of transportation or telecommunication networks, where commodities between pairs of vertices must be routed through the network. Two types of costs are usually considered in NDPs. The first one is the design cost, which is related to the activation or construction of the edges or vertices of the network. The second one is the flow or operational cost, associated with routing the flow through the network. Some NDPs, such as the *Minimum Spanning Tree Problem* [22] and the *Maximum Weight Matching Problem* [13], focus on the design costs. Other NDPs such as the *Shortest Path Problem* [6,12], the *Minimum Cost Flow Problem* [1], and the *Optimum Communication Spanning Tree Problem* [19], focus on the flow costs of the network. A combination of both, design and flow costs has also been considered in several NDPs such as the *Fixed-Charge Network Flow Problem* [32] and the *Multicommodity Network Design Problem* [18].

Among these problems, one of the most intensively studied combinatorial optimization problems is the *Traveling Salesman Problem* (TSP). The TSP consists of determining a Hamiltonian cycle whose design cost is minimized. Many exact and approximate solution techniques have been proposed for this *NP*-hard problem [3].

In this paper we introduce the *Minimum Flow Cost Hamiltonian Cycle Problem* (FCHCP), which can be stated as follows. Given a weighted undirected graph G and a positive flow between pairs of vertices of G, the FCHCP consists of determining a Hamiltonian cycle of G that minimizes the total flow cost between pairs of vertices through their shortest path on the

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cycle. The FCHCP is a combinatorial optimization problem closely related to the TSP. Note that the sets of feasible solutions (Hamiltonian cycles in G) of the TSP and the FCHCP are the same. However, the FCHCP focuses on the flow costs of the network, whereas the TSP focuses on the design costs. Therefore, the sets of optimal solutions of these two problems are not the same because of their difference in the objective functions.

Another combinatorial optimization problem closely related to the FCHCP is the *Optimum Communication Spanning Tree Problem* (OCT). It considers the same operational costs as the FCHCP but its set of feasible solutions is the set of spanning trees. This problem is also known to be *NP*-hard. However, it is important to note that the *Minimum Spanning Tree Problem*, which focuses on the design costs rather than on the flow costs, can be solved in polynomial time with one of several greedy algorithms [1]. The OCT was introduced by Hu in [19] and since then, several approximate solution methods have been developed to solve some instances of the problem i.e. [8,34], and references therein. However, to the best of our knowledge, only one exact solution method has ever been published for this problem [2].

Another problem similar to the FCHCP is the well-known *Quadratic Assignment Problem* (QAP) [21,16], which is considered to be one of the most difficult problems in combinatorial optimization. Instances with 30 vertices were finally solved to optimality for the first time very recently [14,26,27]. In [26,14] the authors also solve to optimality for the first time an instance with 60 and 128 vertices, respectively. The FCHCP is also related to the *Uncapacitated Fixed-Charge Network Flow Problem* (UFC). In particular, if all fixed costs are equal to zero and the solution is a Hamiltonian cycle, then the FCHCP is a particular case of the UFC. Several mathematical models as well as different approximate and exact solution techniques have been proposed for the UFC [18].

Potential applications of the FCHCP arise naturally in telecommunications network design and in rapid transit systems planning. In the former case, cycle topologies are usually preferred when designing reliable networks. If an edge connecting two vertices fails, a cycle topology guarantees the connectivity of the remaining subnetwork and allows the flows to be routed through alternative paths. For these problems it is usually assumed that a forecast of the amount of communication requirements between O/D pairs is available and the objective is to minimize the communication cost after the network has been built (see [36] for an example of data service design). An extensive review of models and telecommunications applications considering the location of a cycle topology is provided in [23]. In the case of rapid transit systems, bus routes and metro lines are sometimes designed with a cycle structure [9] and the objective is to minimize the total travel time to serve users. The design of automated guided vehicles (AGV) networks is yet another relevant transportation application of the FCHCP. The design of AGV networks consists of selecting the optimal route for an automated vehicle that will visit a set of stations within a manufacturing or distribution facility. The network must connect (not necessarily in a direct way) all stations pairs, and the objective is to minimize the total flow cost. For some examples and a review of advantages of a cycle topology in AGV systems, such as the easiness to handle vehicle conflicts and generating a simple network, we refer the reader to [4,5,31].

This paper makes two main contributions. The first one is to introduce and classify a challenging combinatorial optimization problem, referred to as the FCHCP and which, to the best of our knowledge, has never been investigated. We show that this problem is *NP*-hard. Our second contribution is to study the polyhedral structure of the feasible space of the problem. In particular, five different mixed integer programming (MIP) formulations are presented and theoretically compared with respect to the quality of their linear programming (LP) relaxation bounds. We also derive a combinatorial bound and compare its efficiency with respect to the LP bounds of the MIP formulations. Moreover, we introduce some families of valid inequalities to improve the LP bounds of one of the promising MIP formulations. These inequalities are embedded within an exact branch-and-cut algorithm. Finally, we present the results of a series of computational experiments to evaluate the relative performance of some of the proposed mathematical models and of the algorithm.

The remainder of this paper is organized as follows. In Section 2 we formally define the FCHCP and prove that it is *NP*-hard. In Section 3 we introduce five different MIP formulations. In Section 4 we provide a comparison of their LP bounds along with a combinatorial bound. Section 5 describes the proposed valid inequalities and the branch-and-cut algorithm. The results of our computational experiments are presented in Section 6. Conclusions follow in Section 7.

2. Problem definition

Let $G = (V, E)$ be an undirected graph, where V is the set of vertices with $|V| = n \geq 4$, and E is the set of edges $\{i, j\}$. Let A be the set of arcs, where $(i, j), (j, i) \in A$ if and only if $\{i, j\} \in E$. In what follows, a closed non-intersecting sequence of edges is a cycle, a connected sequence of arcs without vertex repetition is a path, and a closed path is a circuit. Also, denote by $c_{ij} \geq 0$ the cost of edge $\{i, j\}$. For each pair of vertices $i, j \in V$, let $w_{ij} \geq 0$ be the amount of flow that must be routed from i to j. For every Hamiltonian cycle H of G and for each pair of vertices $i, j \in V$, let $d_H(i, j)$ denote the cost of the least cost path in H from i to j. The flow cost from i to j in H is thus given by $w_{ij} d_H(i, j)$, and the total flow cost of H is the sum of the flow costs over all pairs of vertices. The FCHCP solution is a Hamiltonian cycle minimizing the total flow cost. It can be stated as

$$\text{(FCHCP)} \qquad \min_{H \subseteq E} \left\{ \sum_{i,j \in V} w_{ij} d_H(i, j) \ : \ H \text{ is a Hamiltonian cycle} \right\}.$$

Example 1. Consider the 4-vertex instance shown in Fig. 1(a), where each edge $\{i, j\}$ is labeled with the values (c_{ij}, w_{ij}). The set of feasible solutions contains three Hamiltonian cycles. For each solution, we compute the least cost path $d_H(i, j)$ for all