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Note

On the Wiener index of generalized Fibonacci cubes and Lucas cubes

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ABSTRACT

The generalized Fibonacci cube $Q_d(f)$ is the graph obtained from the d-cube Q_d by removing all vertices that contain a given binary word f as a factor; the generalized Lucas cube $Q_d(\overleftarrow{f})$ is obtained from Q_d by removing all the vertices that have a circulation containing f as a factor. In this paper the Wiener index of $Q_d(1^s)$ and the Wiener index of $Q_d(\overleftarrow{1^s})$ are expressed as functions of the order of the generalized Fibonacci cubes. For the case $Q_d(111)$ a closed expression is given in terms of Tribonacci numbers. On the negative side, it is proved that if for some d, the graph $Q_d(f)$ (or $Q_d(\overleftarrow{f})$) is not isometric in Q_d, then for any positive integer k, for almost all dimensions d' the distance in $Q_{d'}(f)$ (resp. $Q_{d'}(\overleftarrow{f})$) can exceed the Hamming distance by k.

1. Introduction

The Wiener index of a graph is one of the most studied graph invariants, the main reason for this fact is its vast applicability in theoretical chemistry, cf. the comprehensive surveys [2,3] on the Wiener index of rather specific classes of graphs—trees and hexagonal systems. But this index is also extensively investigated elsewhere, [14,17,21,23] is just a selection of recent papers that indicates a wide variety of topics studied with respect to the Wiener index. Moreover, it is an intrinsic indicator of a potential applicability of (interconnection) networks. In this respect the average distance [1] is more relevant, however the studies of the Wiener index and the average distance are equivalent because for a given graph G, these invariants differ only by the factor $\binom{|V(G)|}{2}$.

In [13] it was demonstrated that each of the Wiener index of Fibonacci cubes and Lucas cubes can be expressed in a closed form. The first of these classes of graphs was introduced as a model for interconnection network [7] and received a lot of attention afterwards, see the survey [11]. Lucas cubes [19] can be considered as a symmetrization of Fibonacci cubes and have found their role in theoretical chemistry [24].

Fibonacci cubes and Lucas cubes were extended to generalized Fibonacci cubes [9] and to generalized Lucas cubes [10], respectively. (We note that the term "generalized Fibonacci cubes" was used in [8] (see also [18,22]) for a restricted family of the graphs from [9].) The main goal of this paper is to extend the results from [13] on the Wiener index of Fibonacci (Lucas)

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cubes to those for generalized Fibonacci (Lucas) cubes that admit isometric embeddings into hypercubes. Such potential classes were identified in [9,10].

We proceed as follows. In the rest of this section we formally introduce the concepts needed in this paper. In the following section the Wiener index of $Q_d(1^s)$ (Theorem 2.3) and the Wiener index of $Q_d(\overleftarrow{1^s})$ (Theorem 2.5) are expressed as sums involving $|V(Q_{d'}(1^s))|$ for some d'. In the case of $Q_d(111)$ it is shown how a closed expression for its Wiener index can be obtained. In the final section we show that if $Q_d(f)$ or $Q_d(\overleftarrow{f})$ is not isometric in Q_d, then in almost all dimensions the distance function is arbitrarily larger than the corresponding Hamming distance.

Graph considered here are finite, simple, and connected. For a (connected) graph G, the *distance* $d_G(u, v)$ (or $d(u, v)$ if G is clear from the context) between vertices u and v is the usual shortest path distance. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. The *Wiener index*, $W(G)$, of a graph G is defined as $\sum d(u, v)$, where the summation runs over all unordered pairs $\{u, v\}$ of vertices of G.

Let $B = \{0, 1\}$ and call the elements of B *bits*. An element of B^d is called a *(binary) word* of length d. We will use the product notation for words meaning concatenation. For example, 1^s0^t is the word of length $s + t$ whose first s bits are 1 and last t bits are 0. A word f is a *factor* of a word u if $u = vfw$ for some words v and w.

The d-cube Q_d is the graph whose vertices are all the binary words of length d, two vertices are adjacent if they differ in exactly one bit. The *Hamming distance* $H(u, v)$ between binary words u and v (of equal length) is the number of bits in which they differ. It is well-known that $d_{Q_d}(u, v) = H(u, v)$ holds for any $u, v \in V(Q_d)$. If f is an arbitrary binary word and d is a positive integer, then the *generalized Fibonacci cube* $Q_d(f)$ is the graph obtained from Q_d by removing all the vertices that contain f as a factor. Similarly, the *generalized Lucas cube* $Q_d(\overleftarrow{f})$ is the graph obtained from Q_d by removing all the vertices that have a circulation containing f as a factor. The *Fibonacci cube* Γ_d is the graph $Q_d(11)$ and the *Lucas cube* Λ_d is $Q_d(\overleftarrow{11})$.

If $b = b_1 \dots b_d$ is a binary word, then let $\bar{b}$ denote its binary complement and let $b^R = b_d \dots b_1$ be the reverse of b. It is easy to see (cf. [9,10]) that if f is an arbitrary binary word, then $Q_d(f) \cong Q_d(\bar{f}) \cong Q_d(f^R)$ and $Q_d(\overleftarrow{f}) \cong Q_d(\overleftarrow{\bar{f}}) \cong Q_d(\overleftarrow{f^R})$, where $\cong$ stands for graph isomorphism. We will implicitly use these facts when considering all possible words.

2. The Wiener index of $Q_d(1^s)$ and $Q_d(\overleftarrow{1^s})$

In this section we extend results from [13] on the Wiener index of $Q_d(11)$ and $Q_d(\overleftarrow{11})$ to $Q_d(1^s)$ and $Q_d(\overleftarrow{1^s})$, respectively. For this sake we will apply the following result from [12] (see also [15] for its wide generalization). If G is a subgraph of Q_d, then set $W_{(i,\chi)}(G) = \{u = u_1 \dots u_d \in V(G) \mid u_i = \chi\}$ for $1 \le i \le d$, $0 \le \chi \le 1$.

Theorem 2.1 ([12]). *If G is an isometric subgraphs of Q_d, then*

$$W(G) = \sum_{i=1}^{d} |W_{(i,0)}(G)| \cdot |W_{(i,1)}(G)|.$$

If $d \ge 1$ and $s \ge 2$, then let $x_d^{(s)} = |V(Q_d(1^s))|$. For any $s \ge 2$ we also set $x_0^{(s)} = 1$ and $x_{-1}^{(s)} = 1$.

Lemma 2.2. *Let $d \ge 1$ and $s \ge 2$. Then $x_d^{(s)} = 2^d$ for $1 \le d \le s - 1$, $x_s^{(s)} = 2^s - 1$, and $x_d^{(s)} = x_{d-1}^{(s)} + x_{d-2}^{(s)} + \cdots + x_{d-s}^{(s)}$ for $d \ge s + 1$.*

Proof. If $d \le s - 1$, then $Q_d(1^s) = Q_d$, hence the first assertion follows. $Q_d(1^s)$ is obtained from Q_d by deleting the vertex 1^s, therefore $x_s^{(s)} = 2^s - 1$. Let now $d \ge s + 1$. Then there are $x_{d-1}^{(s)}$ vertices u of $Q_d(1^s)$ with $u_1 = 0$. The other vertices can be partitioned into those starting with 10 and with 11, respectively. The number of the former ones is $x_{d-2}^{(s)}$, while the other vertices can be partitioned into those starting with 110 and with 111, respectively. Continuing in this manner, and having in mind that 1^s is not a factor of a vertex of $Q_d(1^s)$, the last assertion follows. □

Theorem 2.3. *For any $d \ge 1$ and any $s \ge 2$,*

$$W(Q_d(1^s)) = \sum_{i=1}^{d} \left(x_{i-1}^{(s)} x_{d-i}^{(s)} \left(\sum_{j=2}^{s} x_{i-j}^{(s)} \left(\sum_{k=(d-i-1)-(s-j)}^{d-i-1} x_k^{(s)} \right) \right) \right).$$

Proof. From [9, Proposition 3.1] we know that $Q_d(1^s)$ is an isometric subgraph of Q_d, hence Theorem 2.1 applies to $Q_d(1^s)$.

We first observe that $|W_{(i,0)}(Q_d(1^s))| = x_{i-1}^{(s)} x_{d-i}^{(s)}$ because the factors before and after the ith bit are arbitrary. This assertion also holds for $i = 1$ and for $i = d$ since we have set $x_0^{(s)} = 1$. Consider now the set of vertices $W_{(i,1)}(Q_d(1^s))$ and let u be an arbitrary vertex with $u_i = 1$. Suppose that u_i is preceded with r ones, where $0 \le r \le s - 2$, so that $u_{i-r-1} = 0$. Then the first $i - r - 2$ bits are arbitrary, that is, there are $x_{i-r-2}^{(s)}$ such factors for $0 \le r \le s - 2$. For a fixed r,