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Weak connectedness of tensor product of digraphs

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ABSTRACT

In this paper we give a complete characterization for digraphs whose tensor product is weakly connected, which solves an open problem posed by Harary and Trauth (1966).

1. Introduction

For the notation and terminology below on digraphs, see [1,4,9]. A digraph D is a pair $D = (V(D), E(D))$ with $E(D) \subseteq V(D) \times V(D)$. For an arc $(u, v) \in E(D)$, u is said to be the *initial vertex* of the arc and v the *end vertex*. The *out-degree* (resp. *in-degree*) of a vertex $v \in V(D)$ is defined to be the number of arcs having v as an initial (resp. end) vertex. A *source* (resp. *sink*) is a vertex with in-degree (resp. out-degree) equal to 0.

The *directed cycle* C_n is the digraph with vertex set $[n] = \{1, 2, \dots, n\}$ and arc set $\{(1, 2), (2, 3), \dots, (n, 1)\}$. In particular, C_1 consists of a single vertex with a loop. The *directed path* P_n is C_n with the arc $(n, 1)$ removed. The *length* of a directed cycle or path is defined to be the number of arcs it contains.

A digraph D is called *in-functional* (resp. *out-functional*) if the out-degree (resp. in-degree) of each vertex in D equals to 1. Note that the concept of in-functional digraphs here coincides with the concept of functional digraphs in [5].

For $k \geq 1$, a *route* of length k in D from v_1 to v_{k+1} is a pair $L = (S, \sigma)$, where $S = (v_1, v_2, \dots, v_{k+1})$ is a sequence of $k + 1$ vertices in D, and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ is a sequence consisting of 1 and -1, such that

$$\sigma_i = 1 \Longrightarrow (v_i, v_{i+1}) \in E(D), \quad \text{and} \quad \sigma_i = -1 \Longrightarrow (v_{i+1}, v_i) \in E(D).$$

We call v_1 the initial vertex of L, and v_{k+1} the end vertex. L is said to be *closed* if $v_1 = v_{k+1}$, and *alternating* if $\sigma_i = (-1)^i$ for $1 \leq i \leq k$. The type and weight of L are defined to be σ and $\sum_{i=1}^{k} \sigma_i$ respectively. The greatest common divisor (abbreviated GCD) of the weights of all closed routes in D is called the *weight* of D, where we define $GCD(0, 0) = 0$. Let $w(D)$ denote the weight of D. Then we have $w(D) = 0$ if all closed routes in D have weight 0.

Let $L = (S, \sigma)$ be a route with $S = (v_1, \dots, v_{k+1})$ and $\sigma = (\sigma_1, \dots, \sigma_k)$. Let $R = (T, \tau)$ be another route with $T = (u_1, \dots, u_{l+1})$ and $\tau = (\tau_1, \dots, \tau_l)$. If $v_{k+1} = u_1$, then the *composite route* LR is defined to be $LR = ((v_1, \dots, v_{k+1}, u_2, \dots, u_{l+1}), (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_l))$.

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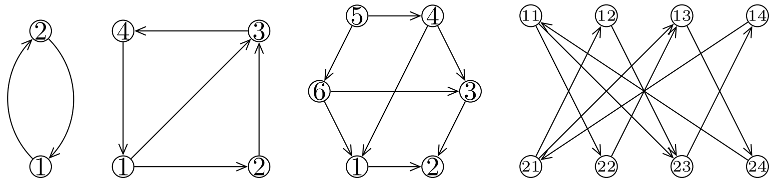

(a) D_1. (b) D_2. (c) D_3. (d) $D_1 \otimes D_2$.

Fig. 1. Examples of weights and tensor product.

A *walk* is a route with type $(1, 1, \ldots, 1)$. There are three kinds of connectedness for digraphs defined as follows. A digraph is said to be *strong* (strongly connected) if for each pair u, v of its vertices, there is a walk from u to v and a walk from v to u. It is *unilateral* (unilaterally connected) if for each pair u, v of its vertices, there is a walk from u to v or a walk from v to u. It is *weak* (weakly connected) if for each pair u, v of its vertices, there is a route from u to v. Note that the digraph P_1 is not weak.

If D is a weak digraph with $w(D) = 0$, then for any $u, v \in V(D)$, all routes in D from u to v have the same weight, which we denote by $w(u, v)$, and we define the *diameter* of D to be $d(D) = max\{w(u, v)|u, v \in V(D)\}$.

Let D_1 and D_2 be digraphs. Their tensor product $D_1 \otimes D_2$ is defined to be the digraph with vertex set $V_1 \times V_2$, and $((u_1, v_1), (u_2, v_2))$ is an arc in $D_1 \otimes D_2$ precisely if (u_1, u_2) and (v_1, v_2) are arcs in D_1 and D_2 respectively. See [8] for more details. Note that we also use the symbol $\otimes$ for the matrix tensor in this paper.

Example 1. We consider the weights of digraphs in Fig. 1. Let $W(D)$ denote the set consisting of weights of closed routes in D with no repeated vertices, other than the repetition of the initial and end vertices. Then we have $W(D_1) = \{0, \pm 2\}$, $W(D_2) = \{0, \pm 1, \pm 3, \pm 4\}$ and $W(D_3) = \{0\}$. Thus $w(D_1) = 2$, $w(D_2) = 1$ and $w(D_3) = 0$. Since $L = ((5, 6, 3, 2), (1, 1, 1))$ is one of the routes in D_3 with maximal weight, we have $d(D_3) = 3$. The digraph in Fig. 1(d) is the tensor product of D_1 and D_2, where we use ij to denote the vertex (i, j).

Considering the connectedness of the tensor product of digraphs, three problems arise naturally: characterize digraphs whose tensor product has each of the three kinds of connectedness. McAndrew [8] solved one of the problems by characterizing digraphs whose tensor product is strong. Harary and Trauth [6] solved the problem for a unilaterally connected tensor product. The problem for a weakly connected tensor product was introduced and analyzed in [6]. Hartfiel and Maxson [7] gave a partial answer to this problem by showing that $D_1 \otimes D_2$ is weak for any weak digraph D_1 if and only if D_2 is chainable. Blondel et al. [2] noticed that this problem is equivalent to characterizing digraphs whose similarity matrices have only positive entries. Despite these contributions on this problem, a convenient characterization for a weakly connected tensor product of digraphs is, to our knowledge, still open.

In this paper, we solve the above problem posed by Harary and Trauth. Let D_1 and D_2 be weak digraphs. In Section 2, we consider the case when D_1 and D_2 are both in-functional or out-functional, and show that $D_1 \otimes D_2$ is weak if and only if the GCD of their weights is 1. In Section 3, we consider the case when D_2 is a directed path of length l, and show that $D_1 \otimes D_2$ is weak if and only if D_1 is l-chainable. In Section 4, we show that the connectedness of the tensor product of two general weak digraphs can be reduced to one of the above two cases, and give a complete characterization for a weakly connected tensor product of digraphs in Theorem 15 and Corollary 16.

2. Tensor product of functional digraphs

Let $L = ((u_1, \ldots, u_k), \sigma)$ and $R = ((v_1, \ldots, v_k), \sigma)$ be routes with the same type in D_1 and D_2 respectively. Then L and R determine bijectively a route $L_0 = (((u_1, v_1), \ldots, (u_k, v_k)), \sigma)$ in $D_1 \otimes D_2$, which we denote by $(u_1, v_1) \xrightarrow{(L,R)} (u_k, v_k)$. We call L and R the projections of L_0 to D_1 and D_2 respectively.

Example 2. Let $L = ((1, 2, 1, 2), (1, 1, -1))$ and $R = ((1, 2, 3, 1), (1, 1, -1))$ be routes in D_1 and D_2 respectively in Fig. 1. Then we have $(1, 1) \xrightarrow{(L,R)} (2, 1) = (((1, 1), (2, 2), (1, 3), (2, 1)), (1, 1, -1))$ being a route from $(1, 1)$ to $(2, 1)$ in $D_1 \otimes D_2$.

The following result gives a necessary condition for a weak tensor product.

Lemma 3. *Let D_1 and D_2 be digraphs with $w(D_1) = d_1$ and $w(D_2) = d_2$. If $D_1 \otimes D_2$ is weak, then $GCD(d_1, d_2) = 1$.*

Proof. Let $u \in V(D_1)$ and $(v_1, v_2) \in E(D_2)$. Since $D_1 \otimes D_2$ is weak, there is a route $(u, v_1) \xrightarrow{(L,R)} (u, v_2)$ from (u, v_1) to (u, v_2) for some routes L and R. Let w be the weight of L. Then L is a closed route in D_1 with weight w, and $R\overleftarrow{v_2 v_1}$ is a closed route in D_2 with weight $w - 1$, where $\overleftarrow{v_2 v_1}$ denotes the route $((v_2, v_1), (-1))$. Since $GCD(w, w - 1) = 1$, the result follows. $\square$

A *directed in-rooted* (resp. *out-rooted*) *tree* is a weak digraph with the out-degree (resp. in-degree) of one distinguished vertex, called the root, equal to 0, and the out-degree (resp. in-degree) of other vertices equal to 1. By [5, Theorem 1], a weak in-functional digraph consists of a directed cycle C_p, called the underlined cycle of the digraph, and some directed in-rooted