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Sparse hypergraphs with applications in combinatorial rigidity[☆]

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ABSTRACT

A hypergraph $H = (V, E)$ is called $(1, k)$-sparse, for some integer k, if each subset $X \subseteq V$ with $|X| \geq k$ spans at most $|X| - k$ hyperedges. If, in addition, $|E| = |V| - k$ holds, then H is $(1, k)$-tight. We develop a new inductive construction of 4-regular $(1, 3)$-tight hypergraphs and use it to solve problems in combinatorial rigidity.

We give a combinatorial characterization of generically projectively rigid hypergraphs on the projective line. Our result also implies an inductive construction of generically minimally affinely rigid hypergraphs in the plane. Based on the rank function of the corresponding count matroid on the edge set of H we obtain combinatorial proofs for some sufficient conditions for the generic affine rigidity of hypergraphs.

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1. Introduction

Given a set of objects (points, lines, bodies, etc.) in $\mathbb{R}^d$ satisfying certain geometric constraints (pairwise distances, directions, incidences, etc.), a basic question is whether (locally or globally) the given constraints uniquely determine the whole configuration up to trivial transformations (rigid motions, dilations, etc.) of the whole set. A well-studied example is the rigidity problem of d-dimensional bar-and-joint frameworks, where the objects are points and the constraints are pairwise distances. In several cases (local or global) uniqueness depends only on the underlying combinatorial structure (for example, the graph of the framework) if the objects are in sufficiently general position.

Our goal is to provide combinatorial tools for attacking such problems in which the underlying combinatorial structure is a hypergraph: projective rigidity, affine rigidity, and scene analysis.

We develop a new inductive construction of 4-regular $(1, 3)$-tight hypergraphs. By using this result we give a combinatorial characterization of generically projectively rigid hypergraphs on the projective line, which was conjectured by George and Ahmed [2]. Our result also implies an inductive construction of generically minimally affinely rigid hypergraphs in the plane. Based on the rank function of the corresponding count matroid on the edge set of H we obtain combinatorial proofs for some sufficient conditions for the generic affine rigidity of hypergraphs, due to Gortler, Gotsman, Liu, and Thurston [3] and Zha and Zhang [9], respectively.

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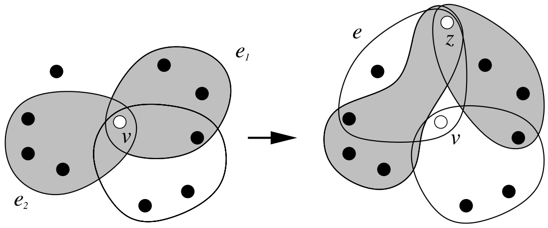

Fig. 1. A 2-extension operation on a 4-uniform hypergraph.

2. Inductive constructions

Let $H = (V, E)$ be a hypergraph and let $X \subseteq V$. We use $i_H(X)$ to denote the number of edges induced by X in H. We say that H is $(1, k)$-*sparse*, for some integer k, if $i_H(X) \leq |X| - k$ for all $X \subseteq V$ with $|X| \geq k$. A $(1, k)$-sparse hypergraph with $|E| = |V| - k$ is called $(1, k)$-*tight*. The hypergraph is called m-*uniform*, for some positive integer m, if each hyperedge $e \in E$ contains exactly m vertices. The degree of a vertex v in H is denoted by $d_H(v)$ and the number of edges of H that contain a given pair $v, w \in V$ is denoted by $d_H(v, w)$. We may omit the subscript referring to H if it is clear from the context.

We introduce a set of operations on $(k + 1)$-uniform hypergraphs which preserve $(1, k)$-sparsity and which can be used to generate all $(k + 1)$-uniform $(1, k)$-tight hypergraphs from a single hyperedge, for all $1 \leq k \leq 3$.

Let $H = (V, E)$ be a $(k + 1)$-uniform hypergraph, let j be an integer with $0 \leq j \leq k - 1$, and let $v \in V$ be a vertex with $d(v) \geq j$. The j-*extension operation at vertex* v picks j hyperedges $e_1, e_2, \ldots, e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size $k + 1$ incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$. Thus the new vertex z has degree $j + 1$ in the extended hypergraph. See Fig. 1. Note that a 0-extension operation simply adds a new vertex z and a new hyperedge of size $k + 1$ incident with z.

The j-extension operation preserves sparsity in the following sense. The simple proof of the next lemma is omitted.

Lemma 2.1. *Let $H = (V, E)$ be a $(k + 1)$-uniform $(1, k)$-sparse ($(1, k)$-tight) hypergraph and let H' be obtained from H by a j-extension operation, where $0 \leq j \leq k - 1$. Then H' is also $(1, k)$-sparse ($(1, k)$-tight, respectively).*

The inverse operation of j-extension can be defined as follows. Let $H = (V, E)$ be a $(k+1)$-uniform hypergraph. Consider a vertex $z \in V$ with $d(z) = j+1$, for some $0 \leq j \leq k-1$, and let v be a neighbour of z in H with $d(z, v) = 1$. Let $e_1, e_2, \ldots, e_{j+1}$ be the edges incident with z, where e_1 is the unique edge which is incident with v, too. The j-*reduction operation at vertex* z *on neighbour* v deletes e_1 and replaces e_i by $e_i - z + v$ for all $2 \leq i \leq j + 1$. Observe that the inverse of j-extension is indeed j-reduction.

We say that a j-reduction operation in a $(k + 1)$-uniform $(1, k)$-sparse hypergraph H is *admissible* if the hypergraph obtained from H by the operation is also $(1, k)$-sparse. To obtain our inductive construction by induction we shall show that each $(k + 1)$-uniform $(1, k)$-sparse hypergraph H (for k up to 3) has a vertex z of degree at most k and that there exists an admissible $(d(z) - 1)$-reduction at z.

Note that the 2-uniform $(1, 1)$-tight hypergraphs are the trees, for which the existence of a vertex of degree one (a leaf) and an admissible 0-reduction (leaf deletion) is straightforward. The case when $k = 2$ is more complicated, but still not very difficult, so we shall omit the proof of this case. Instead, we shall focus on 4-regular $(1, 3)$-tight hypergraphs (see Theorem 2.8 below, which is the main result of this section).

It should also be noted that the above proof strategy does not work when $k \geq 4$. To see this consider the $(1, 4)$-tight 5-uniform hypergraph $H = (V, E)$ with $V = \{v_1, v_2, \ldots, v_7\}$ and $E = \{(v_1, v_2, v_3, v_4, v_7), (v_3, v_4, v_5, v_6, v_7), (v_1, v_2, v_5, v_6, v_7)\}$. We have $d(v_7) = 3$ but each neighbour v_i of v_7 has $d(v_7, v_i) \geq 2$, showing that no 2-reduction can be performed at v_7. Hence an inductive construction for higher values of k is probably more difficult to obtain.

Before dealing with the case of $(1, 3)$-sparse hypergraphs we prove some preliminary lemmas about $(1, k)$-sparse hypergraphs in general. The next lemma is easy to verify by observing that the contribution of a hyperedge to the right hand side of inequality (1) below cannot be less than its contribution to the left hand side.

Lemma 2.2. *Let $H = (V, E)$ be a hypergraph and let $X, Y \subseteq V$ be subsets of vertices. Then*

$$i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y). \quad (1)$$

Let $H = (V, E)$ be a $(k + 1)$-uniform $(1, k)$-sparse hypergraph. We say that a subset $X \subseteq V$ is *critical* if $i(X) = |X| - k$ holds. A subset $Y \subseteq V$ is called *semi-critical* if $i(Y) \geq |Y| - k - 1$.

Lemma 2.3. *Let $H = (V, E)$ be a $(k+1)$-uniform $(1, k)$-sparse hypergraph and let $X, Y \subseteq V$ be subsets of vertices. If $|X \cap Y| \geq k$ and*

(i) *if X and Y are both critical, then $X \cup Y$ is also critical,*
(ii) *if X is critical and Y is semi-critical, then $X \cup Y$ is semi-critical,*

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