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Many-to-many two-disjoint path covers in cylindrical and toroidal grids

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ABSTRACT

A many-to-many k-disjoint path cover of a graph joining two disjoint vertex sets S and T of equal size k is a set of k vertex-disjoint paths between S and T that altogether cover every vertex of the graph. The many-to-many k-disjoint path cover is classified as paired if each source in S is further required to be paired with a specific sink in T, or unpaired otherwise. In this paper, we first establish a necessary and sufficient condition for a bipartite cylindrical grid to have a paired many-to-many 2-disjoint path cover joining S and T. Based on this characterization, we then prove that, provided the set $S \cup T$ contains the equal numbers of vertices from different parts of the bipartition, the bipartite cylindrical grid always has an unpaired many-to-many 2-disjoint path cover. Additionally, we show that such balanced vertex sets also guarantee the existence of a paired many-to-many 2-disjoint path cover for any bipartite toroidal grid even if an arbitrary edge is removed.

1. Introduction

1.1. Many-to-many k-disjoint path covers

In this paper, we consider only a finite simple undirected graph G, whose vertex and edge sets shall be denoted by $V(G)$ and $E(G)$, respectively. For any two vertices u and v in $V(G)$, a path P from u to v in G is a sequence $(w_0, w_1, \ldots, w_n)$ of distinct vertices in $V(G)$ such that $w_0 = u$, $w_n = v$, and w_i and w_{i+1} are adjacent in G for $i \in \{0, \ldots, n-1\}$. If $n \geq 2$ and w_0 and w_n of the sequence are adjacent, the new sequence $(w_0, w_1, \ldots, w_n, w_0)$ is called a cycle. A *path cover* of G is a set of paths in G such that every vertex in $V(G)$ belongs to at least one path. A *vertex-disjoint path cover*, or simply a *disjoint path cover*, of G is a path cover in which every vertex in $V(G)$ is contained in exactly one path.

The major concern of this paper is to study the existence of disjoint path covers for specific classes of graphs, whose paths are additionally required to respectively connect prescribed pairs of distinct vertices. Here, we rephrase the definitions for such constrained disjoint path covers, which were originally given in [37].

Definition 1 (*Paired Many-To-Many k-Disjoint Path Cover*)**.** For a graph G and a positive integer k, let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(G)$. A disjoint path cover $\{P_1, \ldots, P_k\}$ of G is called *a paired many-to-many k-disjoint path cover* if P_i is a path from s_i to t_i for every $i \in \{1, \ldots, k\}$.

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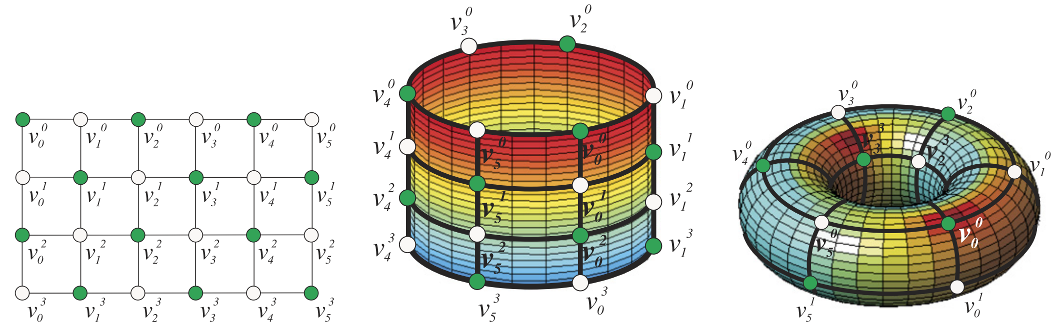

(a) Rectangular. (b) Cylindrical. (c) Toroidal.

Fig. 1. Illustrations of rectangular, cylindrical, and toroidal grids of size 4×6.

Definition 2 (*Unpaired Many-To-Many k-Disjoint Path Cover*)**.** For a graph G and a positive integer k, let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(G)$. A disjoint path cover $\{P_1, \ldots, P_k\}$ of G is called *an unpaired many-to-many k-disjoint path cover* if, for some permutation σ on $\{1, \ldots, k\}$, P_i is a path from s_i to $t_{\sigma(i)}$ for every $i \in \{1, \ldots, k\}$.

Here, the vertices in S and T shall be called *sources* and *sinks*, respectively, and *terminals*, collectively, whose meanings are easily understood. The many-to-many k-disjoint path cover of a graph is a path partition whose k paths respectively join the sources to the sinks. As the definitions say, the *paired* many-to-many k-disjoint path cover is more restrictive than the *unpaired* one in the sense that the paired one is an unpaired many-to-many k-disjoint path cover with the condition that only the identity permutation is used. Two simpler variants of these disjoint path covers have been thought in the literature by allowing a single-vertex terminal set: the *one-to-many* k-disjoint path cover for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ and the *one-to-one* k-disjoint path cover for $S = \{s\}$ and $T = \{t\}$, in which their path covers may share the single-vertex terminal(s) only. (Refer to [21,34,37,41] for more details for these variants.)

The concept of the k-disjoint path cover is naturally associated with that of the vertex-connectivity: Menger's theorem explains the connectivity of a graph in terms of the number of internally vertex-disjoint paths (of type one-to-one) joining two distinct vertices, whereas the Fan lemma describes the connectivity of a graph in terms of the number of internally vertex-disjoint paths (of type one-to-many) joining a vertex to a set of vertices [3]. (Please be aware that, unlike the k-disjoint path cover, the internally vertex-disjoint paths in the theorem and lemma do not have to cover all vertices of the graph.) Moreover, although we do not provide the proof here, it can be shown without a difficulty that a graph is k-connected if and only if it has k vertex-disjoint paths (of type unpaired many-to-many) joining two arbitrary (not necessarily disjoint) vertex sets of size k each, where a vertex that belongs to both sets is considered as a valid, one-vertex path.

The disjoint-path coverability is also closely related to the Hamiltonicity of a graph. For example, a Hamiltonian path between two distinct vertices in a graph forms a many-to-many, one-to-many, and one-to-one 1-disjoint path covers of the graph. A graph is Hamiltonian if and only if it has a one-to-one 2-disjoint path cover for any pair of distinct vertices. Furthermore, a graph of order at least three has a one-to-many 2-disjoint path cover for any terminal sets if and only if it is *Hamiltonian-connected*, that is, every pair of vertices are joined by a Hamiltonian path. Using a disjoint path cover, the construction of a Hamiltonian path or cycle passing through prescribed edges was suggested in [35,37,38]. For the problem of Hamiltonian path or cycle through prescribed edges, also refer to [4,11].

1.2. Rectangular, cylindrical, and toroidal grids

In the context of the Hamiltonian path problem, the rectangular grid graph, or the rectangular grid, first appeared in the literature in [28], which naturally motivated a study on the existence of a 2-disjoint path cover in this graph.

Definition 3 (*Rectangular Grid*)**.** The $m \times n$ *rectangular grid* G is a graph such that $V(G) = \{v_j^i : 0 \le i \le m-1, 0 \le j \le n-1\}$ and $E(G) = \{(v_j^i, v_{j'}^{i'}) : |i - i'| + |j - j'| = 1\}$.

In the formal definition of the $m \times n$ rectangular grid, the vertices are often chosen from the points of the Euclidean plane with integer coordinates so that the vertices and edges form a rectangular grid with n vertices appearing in each of m rows and m vertices in each of n columns (see Fig. 1(a)). In this paper, however, we simply view it as a graph that is isomorphic to the Cartesian product $P_m \times P_n$ of path graphs on m and n vertices because the actual drawing is irrelevant to our study. The rectangular grid is a bipartite graph and thus its vertices may be colored in two colors, green and white, in such a way that every pair of adjacent vertices is colored differently (hereafter, we will denote the color of vertex v by $c(v) \in \{\text{green}, \text{white}\}$).

Besides the rectangular grid graph, we also consider the following two related classes of grid graphs:

Definition 4 (*Cylindrical Grid*)**.** The $m \times n$ *cylindrical grid* G is a graph such that $V(G) = \{v_j^i : 0 \le i \le m-1, 0 \le j \le n-1\}$ and $E(G) = \{(v_j^i, v_{j'}^{i'}) : (j = j' \text{ and } |i - i'| = 1) \text{ or } (i = i' \text{ and } j' \equiv j + 1 \pmod{n})\}$.