Discrete Applied Mathematics 185 (2015) 239–243

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Unique-maximum edge-colouring of plane graphs with respect to faces

Igor Fabrici *, Stanislav Jendrol', Michaela Vrbjarová

Institute of Mathematics, P.J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

ARTICLE INFO

Article history:
Received 22 April 2014
Received in revised form 24 November 2014
Accepted 1 December 2014
Available online 26 December 2014

Keywords:
Plane graph
Edge-colouring
Unique-maximum colouring

ABSTRACT

A unique-maximum k-edge-colouring with respect to faces of a 2-edge-connected plane graph G is an edge-colouring with colours $1, \dots, k$ so that, for each face α of G, the maximum colour occurs exactly once on the edges of α. We prove that any 2-edge-connected plane graph has such a colouring with 3 colours. If we require the colouring to be facially proper then 6 colours are enough.

1. Introduction

Graphs considered in this paper are connected and plane and can have loops and multiple edges. For any graph-theoretical notions not defined here, we refer to [2].

Let G be a 2-edge-connected plane graph with the vertex set $V(G)$, the edge set $E(G)$, and the face set $F(G)$. Faces of G are open 2-cells. The boundary of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices lying in the closure of α that can be organized into a closed walk in the graph G by traversing a simple closed curve just inside the face α. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face α (see [9], p. 101). The *size* of a face $\alpha \in F(G)$, $\deg_G(\alpha)$, is the length of the boundary walk of α. A face α is *k-gonal* if $\deg_G(\alpha) = k$. Two edges of a plane graph are *facially adjacent* if they are adjacent and consecutive on the boundary walk of a face α. Note that in general not any two adjacent edges are facially adjacent.

An *edge-colouring* of G with k colours (*k-edge-colouring*) is a mapping $\varphi : E(G) \to [1, k]$, where $[1, k] = \{1, \dots, k\}$. An edge-colouring φ of a 2-edge-connected plane graph G is *facially proper* if, for any two facially adjacent edges e and f of G, $\varphi(e) \neq \varphi(f)$ holds.

There are lot of recent papers on colourings of elements of plane graphs, where constraints on colourings are given by boundary walks (cycles, trails) of these graphs, see e.g. [8,10], and [15].

Motivations to investigate a unique-maximum k-edge-colouring with respect to faces came from the recent papers [3,4], and [5] where the unique-maximum k-vertex-colourings with respect to paths in graphs and hyperedges in hypergraphs, respectively, are considered. In these papers one can find other motivations and connections to other problems and applications. Note that the unique-maximum vertex-colouring with respect to paths is equivalent to the "ordered colouring", see e.g. [4,11].

* Corresponding author.
E-mail addresses: igor.fabrici@upjs.sk (I. Fabrici), stanislav.jendrol@upjs.sk (S. Jendrol'), michaela.vrbjarova@student.upjs.sk (M. Vrbjarová).

http://dx.doi.org/10.1016/j.dam.2014.12.002
0166-218X/

Let G be a 2-edge-connected plane graph. A *unique-maximum (UM) k-edge-colouring with respect to faces* of G is an edge-colouring with colours $1, \ldots, k$ such that, for each face α of G, the maximum colour occurs exactly once on the edges of the boundary walk of α. (Every edge of a 2-edge connected graph is incident with two different faces, i.e. it occurs at most once on the boundary walk of any face. Every edge of a tree G occurs twice on the boundary walk of the only face of G, thus no colour of this face can be unique. It means, the definition of the UM edge-colouring cannot be used in general for 1-edge-connected graphs.) The minimum k for which G has UM k-edge-colouring is denoted by $\chi'_{\mathrm{um}}(G)$ and the minimum k for which G has a facially proper unique-maximum (FPUM) k-edge-colouring is denoted by $\chi'_{\mathrm{fpum}}(G)$.

The famous Four Colour Problem (4CP) has served as a motivation for many equivalent colouring problems, see e.g. [17]. It was solved in 1976 by Appel and Haken [1] (see also [16] for another proof) and the result is presently known as the Four Colour Theorem 4CT. The 4CT is equivalent to the following (see [17]).

Theorem 1. *The edges of any plane triangulation can be coloured with 3 colours so that the edges bounding an arbitrary face are coloured distinctly.*

Theorem 1 can be reformulated in a form equivalent to the 4CT as follows.

Theorem 2. *Every plane triangulation has a FPUM 3-edge-colouring.*

In general, not all plane graphs are FPUM 3-edge-colourable, as can be seen on the graph W_5 (i.e. the wheel on six vertices), for which $\chi'_{\mathrm{fpum}}(W_5) = 4$.

Hence the following problem can be regarded as a generalization of the 4CP.

Problem. For a given 2-edge-connected plane graph G, determine $\chi'_{\mathrm{fpum}}(G)$.

2. Results

Let G be a plane graph. A set of edges $S \subseteq E(G)$ is *facially independent* if no two edges from S are incident with the same face. There is a simple relation between corresponding sets of edges in a 2-edge-connected plane graph and in its dual.

Lemma 1. *Let G be a 2-edge-connected plane graph and let G^* be the dual of G. If $S^* \subseteq E(G^*)$ is a matching in G^* then the set S of edges of G corresponding to the set S^* in G^* is facially independent.*

Our first result is the following.

Theorem 3. *If G is a 2-edge-connected plane graph then $\chi'_{\mathrm{um}}(G) \leq 3$; moreover, the bound 3 is tight.*

Proof. Consider the dual G^* of the graph G. Let M^* be a maximum matching in G^*. For any vertex u of G^* not covered by M^*, choose an edge that joins u with a vertex v covered by M^*, and denote by S^* the set of so obtained edges. Let M and S be the sets of corresponding edges in G, respectively. Clearly M is a facially independent set in G and the set of faces from S that contain no edge from M is facially independent as well. Colour the edges of M with colour 3, the edges of S with colour 2, and the remaining edges with colour 1. Because $M^* \cup S^*$ covers all vertices of G^*, $M \cup S$ covers all faces of $F(G)$. It is easy to see that so defined edge-colouring fulfils the requirements of the UM edge-colouring. Moreover, the graph $K_4 - e$ (i.e. the complete graph on four vertices minus an edge) is not UM 2-edge-colourable. □

Let G be a 2-edge-connected plane graph. The *medial graph* of G is the graph $M(G)$ with vertex set $E(G)$ in which two vertices are joined by k edges if, in G, they are facially adjacent on k common faces ($k = 0, 1, 2$). The medial graph has a natural planar embedding. Because each edge of G is facially adjacent to four other edges of G (considering the double adjacency on two faces), $M(G)$ is a 4-regular planar graph.

Lemma 2. *Every 2-edge-connected plane graph G has a facially proper 4-edge-colouring.*

Proof. Let G be a 2-edge-connected plane graph and let $M(G)$ be the medial graph of G. By 4CT (or even by Brooks' theorem), $M(G)$ has a proper vertex-colouring with four colours. If we colour edges of G with the colours of the corresponding vertices in $M(G)$, we obtain a colouring in which every two facially adjacent edges of G are assigned different colours. This yields a facially proper edge-colouring of G with four colours. □

Theorem 4. *Let G be a 2-edge-connected plane graph and let G^* be the dual of G. If there exists a matching in G^* covering all vertices of G^* of degree at least 4, then $\chi'_{\mathrm{fpum}}(G) \leq 5$.*

Proof. Let G be a 2-edge-connected plane graph and let G^* be the dual of G. Further, let M^* be a matching of G^* that covers all vertices of G^* of degree at least 4 and let M be the set of corresponding edges in G. Clearly, M is a facially independent set of G (by Lemma 1), and, moreover, every k-gonal face of G, $k \geq 4$, is incident with exactly one edge from M.

According to Lemma 2, there exists a facially proper 4-edge-colouring of G. We recolour all edges from M by colour 5. A new colouring so obtained is facially proper, and the colour 5 occurs exactly once on every k-gonal face of G, $k \geq 4$. Since every k-gonal face, $k \leq 3$, is coloured by k different colours, i.e. every colour is used once, the maximum colour is used exactly once too. □