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The linear chromatic number of a Sperner family

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ABSTRACT

Let S be a finite set and $\mathcal{S}$ a complete Sperner family on S, i.e. a Sperner family such that every $x \in S$ is contained in some member of $\mathcal{S}$. The linear chromatic number of $\mathcal{S}$, defined by Cıvan, is the smallest integer n with the property that there exists a function $f : S \to \{1, \ldots, n\}$ such that if $f(x) = f(y)$, then every set in $\mathcal{S}$ which contains x also contains y or every set in $\mathcal{S}$ which contains y also contains x. We give an explicit formula for the number of complete Sperner families on S of linear chromatic number 2. We also prove tight bounds on the number of elements in a Sperner family of given chromatic number, and prove that complete Sperner families of maximum linear chromatic number are far more numerous those of lesser linear chromatic number.

1. Introduction

Let S be a set. A *Sperner family* on S is an antichain in the poset 2^S; that is, a family $\mathcal{S}$ of subsets of S which are pairwise incomparable by inclusion. For $x \in S$, write $\mathcal{S}_x = \{A \in \mathcal{S} : x \in A\}$; $\mathcal{S}$ is called *complete* if $\bigcup_{A \in \mathcal{S}} A = S$, or equivalently $\mathcal{S}_x \neq \emptyset$ for all $x \in S$. Sperner families have long been of interest in extremal set theory, beginning in the 1930s with the seminal work of Sperner [13] on the maximum size of such a family.

As an adjunct to their study [3] of multicomplexes, Cıvan and Yalçın defined the notion of a *linear coloring* of a simplicial complex. This concept was later generalized by Cıvan [2] in the context of Sperner families: given a set S and integer $k > 0$, a *linear k-coloring* of a Sperner family $\mathcal{S}$ on S is a function $f : S \to \{1, \ldots, k\}$ such that if $f(x) = f(y)$, then $\mathcal{S}_x \subseteq \mathcal{S}_y$ or $\mathcal{S}_y \subseteq \mathcal{S}_x$. This is equivalent to requiring that for each i, $1 \leq i \leq k$, there exist a linear ordering (by refinement) of the set of families $\{\mathcal{S}_v : f(v) = i\}$. The *linear chromatic number* of $\mathcal{S}$, denoted $\lambda(\mathcal{S})$, is the smallest $k > 0$ for which a linear k-coloring exists. For $m > 0$ and $n > 0$, we denote by $ls(m; n)$ the number of complete Sperner families of linear chromatic number n on a set of size m; it is easy to see that $ls(m; 1) = 1$ for all $m > 0$ and that $ls(m; n) = 0$ when $n > m$. We also define $ls(0; 0) = 1$ and $ls(m; 0) = 0$ for $m > 0$.

It is now well known that the number of Sperner families on a set of size m is equal to the number $D(m)$ of monotonic Boolean functions $f : \{0, 1\}^m \to \{0, 1\}$. The computation of $D(m)$ was a problem first posed by Dedekind [5] in 1897, one which has generated a considerable amount of investigation. As discussed in the comprehensive survey [10], various lower and upper bounds for $D(m)$ have been established, and its asymptotic behavior is now quite well understood [9,11]. Nevertheless, exact values for $D(m)$ are still known only for $m \leq 8$. Even though precise formulas and recursive formulas for $D(m)$ have been given (see for example [1,8]), they do not give much information about its value. Thus, the determination of the value of $D(m)$ when $m \geq 9$ is still very much an open problem. Because every nonempty Sperner family on a set of size m

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corresponds to a complete Sperner family on some (uniquely determined) subset, we have the formula:

$$D(m) = 1 + \sum_{k=0}^{m} \sum_{n=0}^{k} ls(k; n).$$

(The extra 1 corresponds to the empty Sperner family and the term $ls(0; 0)$ to the Sperner family whose only member is the empty set.) In view of this relationship, one may view the various numbers $ls(k; n)$ as providing a refinement of the information contained in $D(m)$. In particular, computation (or even estimation) of the former could be helpful as a step towards the larger problem of understanding the latter.

With this goal in mind, the purpose of the present article is to study the numbers $ls(m; n)$, addressing several of the questions raised by Cıvan in [2]. Our first result, proved in Section 3, is an exact formula for $ls(m; 2)$. In Section 4 we study the maximum and minimum size of a complete Sperner family of specified chromatic number, generalizing the well-known theorem of Sperner [13]. Finally, in Section 5, we prove that complete Sperner families of maximum linear chromatic number far outnumber those of lesser chromatic number. This last result suggests that $ls(m; m)$ is the parameter most likely to yield useful information about $D(m)$. The common idea undergirding all the results in this article is that one may study a Sperner family of linear chromatic number m on a set of size n by associating to it to an antichain in the poset $L_{n_1} \times \cdots \times L_{n_m}$, where L_k is the poset $\{0, 1, \ldots, k\}$ with the usual order and $n = n_1 + \cdots + n_m$ is some partition of n into positive integer parts. The technical framework for establishing this correspondence is laid down in Section 2 and is used in each of the subsequent sections.

Throughout this article, we write $Sp_n(S)$ for the set of Sperner families of linear chromatic number n on a set S and $Sp(S) = \cup_n Sp_n(S)$. We also use the notation $[a, b]$ to mean $\{n \in \mathbb{Z} : a \leq n \leq b\}$.

2. Representations of a Sperner family

In this section we describe two convenient ways of representing a Sperner family of specified linear chromatic number. The first involves antichains in chain products and applies to families of arbitrary chromatic number. The second applies only to families of linear chromatic number 2 but is particularly helpful in the solution of the enumeration problem addressed in Section 3.

2.1. Chain products

We begin by defining chain products and establishing some of their basic properties; we then proceed to describe their significance in the context of linear colorings.

In general, a poset P is called *ranked* (or *graded*) if there exists a *ranking function* $r : P \to \mathbb{Z}$ such that $r(y) = r(x) + 1$ whenever y covers x. The ith *level* of P is then defined by $N_i = \{x \in P : r(x) = i\}$. If the ranking function has range $[0, m]$, P is called *rank-symmetric* if $|N_i| = |N_{m-i}|$ for all i and *rank-unimodal* if there exists $\mu \in [0, m]$ such that $|N_i| \leq |N_j|$ for $0 \leq i \leq j \leq \mu$ and $|N_i| \geq |N_j|$ for $\mu \leq i \leq j \leq m$. Finally, P is said to have the *Sperner property* if the maximum size of an antichain in P is equal to the size of the largest level of P.

For any positive integer n, we denote by L_n the usual linear order with underlying set $\{0, 1, \ldots, n\}$. A poset of the form $C_{n_1,\ldots,n_s} = L_{n_1} \times \cdots \times L_{n_s}$, where $n_1, \ldots, n_s$ are positive integers, is called a *chain product*. Note that $C_{n_1,\ldots,n_s}$ has the natural structure of a ranked poset via the function $r(a_1, \ldots, a_s) = a_1 + \cdots + a_s$. We record the following useful fact.

Proposition 2.1 (*[6, p. 230]*). *Chain products are rank-symmetric, rank-unimodal, and have the Sperner property.*

We now describe a natural connection between linear colorings and antichains in chain products. An *ordered partition* of a set S is an n-tuple $\mathcal{P} = (S_1, \ldots, S_n)$ such that $\{S_1, \ldots, S_n\}$ is a partition of S. Given an ordered partition $\mathcal{P}$ as above, a *ranking* of $\mathcal{P}$ is an n-tuple $R = (r_1, \ldots, r_n)$ of bijective functions $r_i : S_i \to \{1, \ldots, |S_i|\}$, $i = 1, \ldots, n$. Now let $M_n(S)$ denote the set of ordered triples $(\mathcal{P}, R, \mathcal{C})$, in which $\mathcal{P} = (S_1, \ldots, S_n)$ is an ordered partition of S, $R = (r_i)_{i=1}^n$ is a ranking of $\mathcal{P}$, and $\mathcal{C}$ is antichain in $C_{|S_1|,\ldots,|S_n|}$. Then there is a function

$$\Phi : M_n(S) \to Sp_n(S)$$

defined as follows: given a triple $(\mathcal{P}, R, \mathcal{C}) \in M_n(S)$, each element $c \in \mathcal{C}$ corresponds to an n-tuple $(c_1, \ldots, c_n)$, where for each i, $0 \leq c_i \leq |S_i|$. Now define $c_S = \bigcup_{i=1}^n \{s_i \in S_i : r_i(s_i) \leq c_i\}$. It is easy to check that $\Phi(\mathcal{P}, R, \mathcal{C}) = \{c_S : c \in \mathcal{C}\}$ is a Sperner family in S and that the coloring $\lambda : S \to \{1, \ldots, n\}$ defined by sending $s \in S$ to the (unique) i such that $s \in S_i$ is an n-linear coloring of this Sperner family.

Proposition 2.2. *Φ is surjective; moreover, if $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$, then $|\mathcal{C}| = |\mathcal{S}|$.*

Proof. Let $\mathcal{S}$ be an n-linearly colorable Sperner family on S. Then there is a partition of S into color classes $S_1, \ldots, S_n$, and for $i = 1, \ldots, n$, there exists a bijection $r_i : S_i \to [1, \ldots, |S_i|]$ such that $\mathcal{S}_x \subseteq \mathcal{S}_y$ if $r_i(y) \leq r_i(x)$. Now consider the element of $M_n(S)$ described by the partition $\mathcal{P} = (S_1, \ldots, S_n)$, the ranking $R = (r_i)_{i=1}^n$, and the antichain $\mathcal{C} = \{C(A) : A \in \mathcal{S}\}$ in $C_{|S_1|,\ldots,|S_n|}$, where $C(A) = (c_1, \ldots, c_n)$ is defined by $c_i = \max\{0\} \cup r_i(A \cap S_i)$. Clearly $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$, and the equality $|\mathcal{C}| = |\mathcal{S}|$ is obvious from the construction. $\square$