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A linear algorithm for secure domination in trees

A.P. Burger [a], A.P. de Villiers [a], J.H. van Vuuren [b,*]

[a] Department of Logistics, Stellenbosch University, Private Bag X1, Matieland, 7602, South Africa

[b] Department of Industrial Engineering, Stellenbosch University, Private Bag X1, Matieland, 7602, South Africa

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ABSTRACT

A subset X of the vertex set of a graph G is a *secure dominating set* of G if X is a dominating set of G and if, for each vertex u not in X, there is a neighbouring vertex v of u in X such that the swap set $(X - \{v\}) \cup \{u\}$ is again a dominating set of G. The *secure domination number* of G, denoted by $\gamma_s(G)$, is the cardinality of a smallest secure dominating set of G. A linear algorithm is proposed in this paper for finding a minimum secure dominating set and hence the value $\gamma_s(T)$ for a tree T. The algorithm is based on a strategy of repeatedly pruning away pendent spiders of T after having dominated them securely.

1. Introduction

Let $G = (V, E)$ be a simple graph of order n. A set $X \subseteq V$ is a *dominating set* of G if every vertex in $V - X$ is adjacent to some vertex in X. A vertex $v \in X$ is *defended* by itself, while a vertex $u \notin X$ is *defended* by an adjacent vertex $v \in X$ if the *swap set* $(X - \{v\}) \cup \{u\}$ is again a dominating set of G. A set $X \subseteq V$ is a *secure dominating set* (SDS) of G if it is a dominating set of G and if every vertex of G is defended. An SDS of G of minimum cardinality is called a *minimum secure dominating set* (MSDS) of G and this minimum cardinality is denoted by $\gamma_s(G)$, called the *secure domination number* of G. Various properties of SDSs and bounds on $\gamma_s(G)$ have been established for various graph classes in the literature [1,2,4–6,8,11,13].

There are numerous applications of the notion of secure domination. If the vertices of G denote locations in some spatial domain, and the edges model links between these locations along which patrolling guards may move, then a secure dominating set of G represents a set of locations at which guards may be stationed so that the entire location complex modelled by G is protected in the sense that if a security concern arises at location u, there will either be a guard stationed at that location who can deal with the problem, or else a guard dealing with the problem from an adjacent location v will still leave the location complex dominated after moving from location v to location u in order to deal with the problem. In this scenario the secure domination number represents the minimum number of guards required to protect the entire location complex in a bid to save on the total guard deployment cost. The above generic application is often realised in the context of surveillance applications, military strategy analysis or the deployment of security guards by private security firms.

The decision problem associated with finding a minimum dominating set of an arbitrary graph is **NP**-complete [10, p. 75] and so is the decision problem associated with finding an MSDS of an arbitrary graph [9]. Perhaps the fastest exponential-time algorithm for finding a dominating set of minimum cardinality in an arbitrary graph was proposed by Van Rooij and Bodlaender [14] (in the guise of an algorithm for the celebrated set cover problem), while two exponential-time algorithms (a branch-and-bound algorithm and a branch-and-reduce algorithm) for finding an MSDS of an arbitrary graph were proposed

* Corresponding author. Tel.: +27 21 808 4244; fax: +27 21 808 4245.
E-mail addresses: apburger@sun.ac.za (A.P. Burger), antondev@sun.ac.za (A.P. de Villiers), vuuren@sun.ac.za (J.H. van Vuuren).

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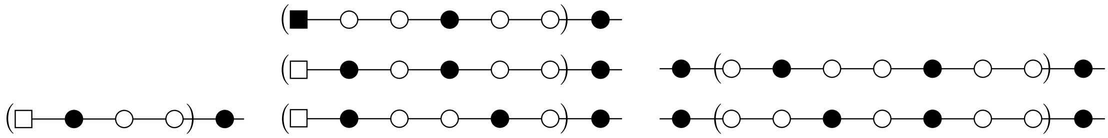

(a) $j = 4$. (b) $j = 6$. (c) $j \geq 7$.

Fig. 2.1. Solid vertices form part of the dominating sets of cardinality $\lceil 3(j-1)/7 \rceil - 1$ for the subpaths in brackets. Leaves are denoted by square vertices. None of these dominating sets is an SDS.

by Burger et al. [3]. Cockayne et al. [7], however, designed a linear-time algorithm for domination in trees which has inspired us to pursue a linear algorithm for finding MSDSs of trees in this paper. The advantage of having a fast algorithm for computing MSDSs of trees is that it may be used to determine an upper bound on the secure domination number $\gamma_s(G)$ of an arbitrary (connected) graph G, by applying the algorithm to any spanning tree of G.

After reviewing and establishing a number of preliminary results in Section 2, we describe in Section 3 how an MSDS may be found for a spider (a tree with at most one vertex of degree at least 3). This description serves as the foundation on which we base the design of a more general algorithmic approach in Section 4 for finding an MSDS of any tree. Finally, Section 5 contains a description of how the algorithmic approach of Section 4 may be implemented in linear time and space. To our best knowledge the algorithm in this paper is the first polynomial algorithm for finding MSDSs of trees.

2. Preliminary results

Any vertex of degree at least 3 in a tree T is called a *branch vertex* of T. If T contains no branch vertex, then it is a path. The following result, dating from 2005, holds for paths and is due to Cockayne et al. [8].

Lemma 1 ([8]). *If T is a path $\mathscr{P}_n$ of order n, then $\gamma_s(\mathscr{P}_n) = \lceil 3n/7 \rceil$.*

An *endpath* $\mathscr{P}$ of a tree T is a subpath of T that contains a leaf ℓ of T and in which every vertex $v \neq \ell$ has degree 2 in T. The following result is central to the development of our algorithm.

Lemma 2. *Let T be a tree and let $\mathscr{P}$ be an endpath of order j in T. Then there is no SDS of T containing fewer than $\lceil 3(j-1)/7 \rceil$ vertices of $\mathscr{P}$.*

Proof. For $1 \leq j \leq 3$ the quantity $\lceil 3(j-1)/7 \rceil - 1$ is non-positive. For $j = 4$ there is only one possible dominating set of cardinality 1 for $\mathscr{P}$, and then only in the best-case scenario where the vertex immediately outside $\mathscr{P}$ in T is included in the dominating set, as shown in Fig. 2.1(a). For $j = 5$, there is not even a dominating set of cardinality 1 for $\mathscr{P}$, let alone an SDS. For $j = 6$ there are three possible dominating sets of cardinality 2 for $\mathscr{P}$ (again in the best-case scenario where the vertex immediately outside $\mathscr{P}$ in T is included in the dominating set), as shown in Fig. 2.1(b). Since none of these dominating sets is an SDS of $\mathscr{P}$, the statement is true for $j \leq 6$.

Furthermore, any stretch of seven consecutive vertices in $\mathscr{P}$ requires at least three vertices in any secure dominating set of $\mathscr{P}$ for $j \geq 7$. This may be seen by noting that neither of the only two dominating sets of cardinality 2 for the subpath of order 7 in Fig. 2.1(c) securely dominates the subpath, even in the best-case scenario where the two vertices immediately outside the subpath are both included in the dominating set.

Now suppose $j = 7s + t$ for some integer $0 \leq t < 7$. Then it follows that at least

$$\begin{aligned}\left\lceil \frac{3(t-1)}{7} \right\rceil + 3s &> \left\lceil \frac{3(t-1)}{7} \right\rceil - 1 + 3s \\ &= \left\lceil \frac{3(7s+t-1)}{7} \right\rceil - 1 \\ &= \left\lceil \frac{3(j-1)}{7} \right\rceil - 1\end{aligned}$$

vertices are required to dominate $\mathscr{P}$ securely. ■

3. Spiders

The notion of a spider[1] plays an important role in our algorithmic approach towards determining an MSDS of an arbitrary tree. A *spider* $S = S(a_1, \ldots, a_r)$ is a tree formed by joining $r \geq 1$ vertex-disjoint paths of orders $a_1, \ldots, a_r$ as pendent paths

[1] This special kind of tree is sometimes also called a *wounded spider*.