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Vulnerability of nearest neighbor graphs

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ABSTRACT

We study several measures of vulnerability of nearest neighbor graphs. We obtain estimates on the toughness, the tenacity, the scattering number, and the rupture degree of nearest neighbor graphs. The method is to recursively apply a certain separator theorem an appropriate number of times, depending on the particular measure studied. We also apply these methods to obtain estimates of these measures for several other classes of graphs.

1. Introduction

Graphs equipped with small separators can be divided into two pieces of roughly equal size by removing a small number of vertices. As such, separator theorems are useful for implementing "divide and conquer" techniques to design efficient algorithms. Several natural families of graphs such as planar graphs and nearest neighbor graphs have small separators [17,19]. However, graphs with small separators would model inefficient communications networks. Here we examine the vulnerabilities of such graphs more precisely by studying specific measures of connectedness such as the integrity, the toughness, the tenacity, the scattering number, and the rupture degree. The unifying concept is that all of these measures are defined using various combinations of the same parameters. These parameters can be estimated by successively applying a separator theorem. As a consequence, estimates of these measures can be obtained for certain classes of graphs.

The integrity $I(G)$ of a simple graph G was introduced by Barefoot, Entringer, and Swart [6]. The integrity, in a sense, measures the effort needed to shatter a graph into small pieces. In [8] the integrity of certain families of graphs was studied. It was demonstrated there that for a planar graph G with n vertices, we have $I(G) \in O(n^{2/3})$ and that $2/3$ is the best possible exponent. A nearest neighbor graph is a geometric graph whose vertex set is associated to points in a metric space (generally $\mathbb{R}^d$). Thus, to any pair of vertices we can attach a distance between the pair. Two vertices in a nearest neighbor graph of type k are adjacent if and only if one of the vertices is among the k nearest neighbors of the other vertex. Nearest neighbor graphs possess a separator theorem [19] and the connectedness of certain random nearest neighbor graphs has been well-studied [5]. Here we extend the methods of [8] and apply them to nearest neighbor graphs and several other classes of graphs. The same methods are also used to examine the toughness $t(G)$ [7], the tenacity $T(G)$ [9], the scattering number $\text{sc}(G)$ [14,21], and the rupture degree $r(G)$ [16] of such graphs. The precise definitions, and some motivation, for these measures are given in Section 3. All graphs here are considered simple and we set $n = |V(G)|$.

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We are interested in asymptotic estimates of the relevant measures. In the following result, all of the constants can be made explicit from Section 4.

Theorem 1.1. *Let G be a k-type nearest neighbor graph embedded in $\mathbb{R}^d$. Then for sufficiently large n there are positive numbers a_d, b_d, and c_d depending only on d so that*

$$I(G) \le a_d c_d^{d/(d+1)} k^{1/(d+1)} n^{d/(d+1)} - b_d c_d k^{1/d} n^{(d-1)/d},$$
$$t(G) \le 2^{(d^2-1)/d} b_d^d c_d^d k - 2^{d+1} b_d^{d+1} c_d^{d+1} k^{(d+1)/d} n^{-1/d},$$
$$T(G) \le 2^{(d^2-1)/d} b_d^d c_d^d k - 2^{d+1} b_d^{d+1} c_d^{d+1} k^{(d+1)/d} n^{-1/d} + 2^{2d+2} b_d^{2d} c_d^{2d} k^2 n^{-1},$$
$$\mathrm{sc}(G) \ge 2 - c_d k^{1/d} n^{(d-1)/d},$$
$$r(G) \ge -a_d c_d^{d/(d+1)} k^{1/(d+1)} n^{d/(d+1)} + b_d c_d k^{1/d} n^{(d-1)/d} + \sqrt{2} d^{-d/(d+1)} b_d^{-d/(d+1)} c_d^{-d/(d+1)} k^{-1/(d+1)} n^{1/(d+1)}.$$

The numbers a_d, b_d, and c_d are given explicitly at the beginning of Section 4. Note that as $k < n$ we have $k^{1/(d+1)} n^{d/(d+1)} > k^{1/d} n^{(d-1)/d}$ and that $t(G), T(G) \in O(k)$. How large n needs to be depends on d and k and can be made precise as indicated in the proof of Theorem 3.2 in Section 3. As an example, for G as above with $d = 3$ we compute $a_3 = 4.925$, $b_3 = 3.847$ and $c_3 = 29.394$ from Section 4. Then for sufficiently large n we have the following explicit estimates,

$$I(G) \le 45.52 k^{1/4} n^{3/4} - 74.61 k^{1/3} n^{2/3},$$
$$t(G) \le 9180985.13k - 2616034622 k^{4/3} n^{-1/3},$$
$$T(G) \le 9180985.13k - 2616034622 k^{4/3} n^{-1/3} + 5.36 \times 10^{14} k^2 n^{-1},$$
$$\mathrm{sc}(G) \ge 2 - 19.39 k^{1/3} n^{2/3},$$
$$r(G) \ge -45.52 k^{1/4} n^{3/4} + 74.61 k^{1/3} n^{2/3} + 0.02 k^{-1/4} n^{1/4}.$$

In Section 2 we give a brief catalog of various separator theorems, and following [1], we prove a precise version of the separator theorem for nearest neighbor graphs from [19]. In Section 3 we show that a separator theorem of a certain type implies bounds on the measures of vulnerability considered here. This result is used to prove Theorem 1.1 in Section 4. There, we also apply the results of Section 3 to several other families of graphs such as planar graphs, graphs of fixed genus, and minor-excluded graphs.

2. Separator theorems

For $A \subset V(G)$ we denote by $G[A]$ the maximal subgraph of G with vertex set A, and we say that $G[A]$ is the subgraph induced by A. Let $G \setminus A$ denote $G[V(G) \setminus A]$.

Definition 2.1. Let f be a function so that $f(x) \le x$ and let $\delta \in [1/2, 1)$. A set $C \subset V(G)$ with $|V(G)| = n$ is an $f(n)$ separator that δ-splits G if $|C| \le f(n)$ and $V(G \setminus C) = A \sqcup B$ so that $|A|, |B| \le \delta n$ and there are no edges from A to B.

The following have been previously found as separator theorems for various classes of graphs. If G is a planar graph with n vertices then G has a $\frac{3\sqrt{2}}{2}\sqrt{n}$ separator that 2/3-splits G, see [4,17]. For G a graph of genus g with n vertices, then according to [10,11], G has a $\sqrt{6(2g+1)n}$ separator that 2/3-splits G. A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. An H-minor of G is a minor of G that is isomorphic to H. If G is a graph on n vertices with no K_h-minor, then G has a $h^{3/2}\sqrt{n}$ separator that 2/3-splits G, see [3,2]. This has been improved in the h-aspect in [15] to a $ch\sqrt{n}$ separator. However, the constant c in the bound for this separator is not given precisely, and depends on several subtle graph parameters. For chordal graphs with n vertices and m edges, G has a $c\sqrt{m}$ separator that 1/2-splits G, for some constant c, [12]. In [19,20,23] it was shown that a k-type nearest neighbor graph in $\mathbb{R}^d$ has an $f(n)$ separator that $\frac{d+1}{d+2}$-splits the graph, where $f \in O(k^{1/d} n^{(d-1)/d})$. In this section we make this last separator theorem more precise.

The following result is essentially Theorem 8.5 in [1] which is stated, but not proved, and without giving the constant for the bound of the separator. We give a proof of the result to obtain a precise constant for the separator and also for the sake of completeness. In the sequel, $\Gamma(s)$ is the gamma function and τ_d is the kissing number in d dimensions, [19]. That is, τ_d is the maximum number of unit balls in $\mathbb{R}^d$, with nonoverlapping interiors, that can be arranged so that they are all tangent to another unit ball. The kissing number depends only on d and $\tau_1 = 2$, $\tau_2 = 6$, and $\tau_3 = 12$ for example. The kissing number for general n is difficult to determine precisely, but asymptotic upper and lower bounds are known.

Theorem 2.2. *Let G be a nearest neighbor graph in $\mathbb{R}^d$ of type k. Then G has a*

$$2d^{1/d}\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)}\right)^{(d-1)/d} \tau_d^{1/d} k^{1/d} n^{(d-1)/d} \qquad (1)$$

separator that $\frac{d+1}{d+2}$-splits G.