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Dynamic monopolies in directed graphs: The spread of unilateral influence in social networks

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ABSTRACT

Irreversible dynamic monopolies arise from the formulation of the irreversible spread of influence such as disease, opinion, adaptation of a new product, etc., in social networks. In some applications, the influence in the underlying network is unilateral or one-sided. In order to study the latter models we need to introduce the concept of dynamic monopolies in directed graphs. Let G be a directed graph such that the in-degree of any vertex of G is at least one. Let also $\tau : V(G) \to \mathbb{N}$ be an assignment of thresholds to the vertices of G. A subset M of vertices of G is called a dynamic monopoly for (G, τ) if the vertex set of G can be partitioned into $D_0 \cup \cdots \cup D_t$ such that $D_0 = M$ and for any $i \geq 1$ and any $v \in D_i$, the number of edges from $D_0 \cup \cdots \cup D_{i-1}$ to v is at least $\tau(v)$. One of the most applicable and widely studied threshold assignments in directed graphs is strict majority threshold assignment in which for any vertex v, $\tau(v) = \lceil (\deg^-(v) + 1)/2 \rceil$, where $\deg^-(v)$ stands for the in-degree of v. In this paper we first discuss some basic upper and lower bounds for the size of dynamic monopolies with general threshold assignments and then obtain some hardness results concerning the smallest size of dynamic monopolies in directed graphs. We prove that any strongly connected directed graph G admits a strict majority dynamic monopoly with at most $\lceil |G|/2 \rceil$ vertices. Next we show that any simple directed graph on n vertices and with positive minimum in-degree admits a strict majority dynamic monopoly with at most $n/2$ vertices, where by a simple directed graph we mean any directed graph $G = (V, E)$ such that $(u, v) \in E$ implies $(v, u) \notin E$ for all $u, v \in V$. We show that this bound is achieved by a polynomial time algorithm. This upper bound improves greatly the previous best known result. The final note of the paper deals with the possibility of the improvement of the latter $n/2$ bound.

1. Introduction and motivation

The irreversible spread of influence in social networks such as spread of disease, of opinion etc. is modeling in terms of progressive (or irreversible) dynamic monopolies in undirected graphs. In this formulation the elements of the network are represented by the nodes of a graph $G = (V, E)$ and the links of the network by the edges of G. Assume that corresponding to any vertex v of G an integer value denoted by $\tau(v)$ is given. This value is called the threshold of v and the assignment $v \to \tau(v)$ is called a threshold assignment of G. Let a graph G and an assignment of thresholds τ to its vertices be given. By a τ-dynamic monopoly we mean any subset D of G such that the vertex set of G can be partitioned into subsets $D_0, D_1, \ldots, D_k$

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such that $D_0 = D$ and for any $i = 1, \dots, k-1$ each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \cdots \cup D_i$. Irreversible dynamic monopolies were widely studied in the literature in [2,6,7,12,14,13,17,18], under the equivalent term "conversion sets" [5,11] and also "target set selection" [1,9,15]. Dynamic monopolies have applications in viral marketing [10]. A concept similar to dynamic monopolies, the so-called bootstrap percolation was widely studied in the area of percolation theory (see e.g. [3]). In such setup the threshold assignment is constant for all vertices of the graph. Different kinds of threshold assignments such as constant assignments, simple majority assignment, where for any vertex v, $\tau(v) = \lceil \deg(v)/2 \rceil$ and strict majority assignment, where for any vertex v, $\tau(v) = \lceil (\deg(v)+1)/2 \rceil$ were defined and studied in these researches. Dynamic monopolies in terms of the average threshold were studied in [13]. In [7,14] the authors have studied dynamic monopolies of random graphs. Dynamic monopolies with probabilistic thresholds was also studied in [17]. A relationship between dynamic monopolies and degeneracy of graphs was obtained in [18].

The usual formulation of dynamic monopolies is in terms of a discrete time dynamic process defined as follows. Consider a discrete time dynamic process on the vertices of G, where some vertices of G are considered as active vertices at the beginning of the process i.e. at time zero (activeness is interpreted according to the underlying phenomenon such as disease, opinion, etc.). Denote the set of active vertices at any discrete time $t \geq 0$ by D_t. Assume that at the beginning of the process (i.e. at time zero), the vertices of a subset $D \subseteq V(G)$ are active. Hence $D_0 = D$. At each discrete time i any inactive vertex v is activated provided that v has at least $\tau(v)$ active neighbors in $D_0 \cup \cdots \cup D_{i-1}$. If at the end of the process all vertices are active then the starting subset D is called a *dynamic monopoly*. Some well-known threshold assignments for the vertices of a graph G are simple majority threshold, where for any vertex v, $\tau(v) = \lceil \deg(v)/2 \rceil$ and strict majority threshold, where $\tau(v) = \lceil (\deg(v)+1)/2 \rceil$.

While formulating the spread of influence by undirected graphs it is assumed that the influence is a mutual property i.e. when a vertex v does influence another vertex u then u too does influence v. We notice that in some applications influence is a unilateral or one-sided relationship. For instance a person may have an influential role to another person but does not effect from the same person. For such models we have to use directed graphs and extend the concept of dynamic monopolies for directed graphs. Throughout this paper we consider simple directed graphs. A directed graph $G = (V, E)$ is simple if it contains no loop and there exists at most one edge between any two vertices of G. In particular, there exists no directed cycle of length two in G. We refer the reader for other concepts concerning directed graphs not defined in this paper to [16]. Although we consider simple directed graphs, some of our theorems are still valid for multiple directed graphs (e.g. Theorem 7). We make a remark on this point at the concluding remarks of the paper. We present the following formal definition.

Definition 1. Let G be a directed graph such that the in-degree of any vertex of G is at least one. Let also $\tau : V(G) \to \mathbb{N}$ be an assignment of thresholds to the vertices of G such that $\tau(v) \leq \deg^-(v)$, for any vertex v, where $\deg^-(v)$ stands for the in-degree of v. A subset M of vertices of G is called a dynamic monopoly for (G, τ) if the vertex set of G can be partitioned into $D_0 \cup \cdots \cup D_t$ such that $D_0 = M$ and for any $i \geq 1$ and any $v \in D_i$, the number of edges from $D_0 \cup \cdots \cup D_{i-1}$ to v is at least $\tau(v)$.

For any two vertices u and v if there is an edge from u to v then we say u is an in-neighbor of v. Let us remark that since in this model any vertex can only be affected by its in-neighbor vertices then it is assumed that all directed graphs in this paper have positive minimum in-degree. We denote the order of G by $|G|$.

Two special types of threshold assignments are mostly studied in the area of dynamic monopolies both in directed and undirected graphs. Let G be a directed graph by the simple (resp. strict) majority threshold for G we mean the threshold function τ such that $\tau(v) = \deg^-(v)/2$ (resp. $\tau(v) = \lceil (\deg^-(v)+1)/2 \rceil$) for any vertex v of G, where $\deg^-(v)$ stands for the in-degree of v. By a strict majority dynamic monopoly for a graph G we mean any dynamic monopoly for G with strict majority threshold assignment. Strict majority dynamic monopolies were widely studied in the literature. First in [6], Chang and Lyuu have obtained the upper bound $23|G|/27$ for the smallest size of strict majority dynamic monopoly in any general directed graph G. Then the same authors improved this bound to $0.7732|G|$ in [7]. Recently this bound has been improved to $2|G|/3$ in [8] and by a very shorter proof in [1] by Ackerman et al. We show in Section 2 of this paper that the smallest size of strict majority dynamic monopoly in any simple directed graph G is at most $|G|/2$. The majority and strict majority dynamic monopolies of undirected graphs were already studied by the authors in [13].

The outline of the paper is as follows. In the rest of this section we discuss an upper and a lower bound for the size of dynamic monopolies with general thresholds. Then in Section 2, we obtain some hardness results concerning the complexity status of determining the smallest size of dynamic monopolies with strict majority threshold and with constant threshold assignment $\tau(v) = 2$. Next in Section 3, we first show that any strongly connected directed graph admits a strict majority dynamic monopoly with at most $\lceil |G|/2 \rceil$ vertices (Theorem 7). Then we reduce the latter bound to $\lfloor |G|/2 \rfloor$ (Theorem 8). In fact to prove this bound we need the proof of the upper bound $\lceil |G|/2 \rceil$. Finally using this result we show that any simple directed graph G contains a strict majority dynamic monopoly with at most $\lfloor |G|/2 \rfloor$ vertices (Theorem 10). Such a strict majority dynamic monopoly can be obtained by a polynomial time algorithm (Remark 3). At the last section we first show that the upper bound of Theorem 10 cannot be improved to any bound better than $(2/5)|G|$, i.e. to any bound with order of magnitude $(2/5)|G| - o(1)$. We end the paper with mentioning an open question about the smallest size of strict majority dynamic monopolies.

For directed graphs with general thresholds we have the following interesting result from [1]. Recall that the in-degree of any vertex v is denoted by $\deg^-(v)$.