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An edge-separating theorem on the second smallest normalized Laplacian eigenvalue of a graph and its applications☆

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ABSTRACT

Let $\lambda_2(G)$ be the second smallest normalized Laplacian eigenvalue of a graph G. In this paper, we investigate the behavior on $\lambda_2(G)$ when the graph G is perturbed by separating an edge. This result can be used to determine all trees and unicyclic graphs with $\lambda_2(G) \geq 1-\frac{\sqrt{2}}{2}$. Moreover, the trees and unicyclic graphs with $\lambda_2(G) = 1-\frac{\sqrt{2}}{2}$ are also determined, respectively.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. For $v \in V(G)$, let $d(v)$ be the degree of v. The maximum and minimum degrees of G are denoted by Δ and δ, respectively. The diameter of G is denoted by $\text{diam}(G)$. We use the notation I for the identity matrix, $\boldsymbol{e}$ for the vector consisting of all ones, M^T for the transpose of a real matrix M.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian and normalized Laplacian matrices of G are defined as $L(G) = D(G) - A(G)$ and $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$, respectively. For $v \in V(G)$, let $\mathcal{L}_v(G)$ be the principal submatrix of $\mathcal{L}(G)$ formed by deleting the row and the column corresponding to the vertex v. When only one graph G is under consideration, we sometimes use A, D, L, $\mathcal{L}$ and $\mathcal{L}_v$ instead of $A(G)$, $D(G)$, $L(G)$, $\mathcal{L}(G)$ and $\mathcal{L}_v(G)$, respectively. It is easy to see that $\mathcal{L}(G)$ is a symmetric positive semidefinite matrix and $D(G)^{1/2}\boldsymbol{e}$ is an eigenvector of $\mathcal{L}(G)$ with eigenvalue 0. Thus, the eigenvalues $\lambda_i(G)$ $(1 \leq i \leq n)$ of $\mathcal{L}(G)$ (or the normalized Laplacian eigenvalues of G) satisfy

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

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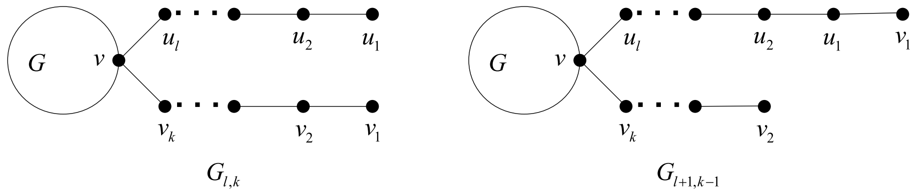

Fig. 1. Grafting an edge.

Some of them may be repeated according to their multiplicities. We call $\lambda_k(G)$ the k-th smallest normalized Laplacian eigenvalue of G. When only one graph G is under consideration, we sometimes write λ_k instead of $\lambda_k(G)$, for $1 \le k \le n$.

Chung [3] showed that the second smallest normalized Laplacian eigenvalue $\lambda_2(G)$ is 0 if and only if G is disconnected. Thus $\lambda_2(G)$ is popularly known as a good parameter to measure how well a graph is connected. It is also closely related to the discrete Cheeger's constant, isoperimetric problems, etc. (see [3]).

Let $\boldsymbol{g}$ be an eigenvector of $\mathcal{L}(G)$. Then we can view $\boldsymbol{g}$ as a function which assigns to each vertex v of G a real value $g(v)$. By letting $\boldsymbol{g} = D^{1/2}\boldsymbol{f}$, we have

$$\frac{\boldsymbol{g}^T\mathcal{L}\boldsymbol{g}}{\boldsymbol{g}^T\boldsymbol{g}} = \frac{\boldsymbol{f}^TD^{1/2}\mathcal{L}D^{1/2}\boldsymbol{f}}{(D^{1/2}\boldsymbol{f})^TD^{1/2}\boldsymbol{f}} = \frac{\boldsymbol{f}^TL\boldsymbol{f}}{\boldsymbol{f}^TD\boldsymbol{f}} = \frac{\sum\limits_{uv\in E(G)}(f(u)-f(v))^2}{\sum\limits_{v\in V(G)}d(v)f(v)^2}.$$

Thus, the following formula for $\lambda_2(G)$ is clear.

$$\lambda_2(G) = \inf_{\boldsymbol{f}\perp D\boldsymbol{e}} \frac{\sum\limits_{uv\in E(G)}(f(u)-f(v))^2}{\sum\limits_{v\in V(G)}d(v)f(v)^2}. \tag{1.1}$$

A vector $\boldsymbol{f}$ that attains the infimum on the right side of (1.1) is called a harmonic eigenfunction associated with $\lambda_2(G)$.

The basic properties on $\lambda_2(G)$ are listed as follows.

Proposition 1.1 ([3]). *Let G be a graph and $\boldsymbol{f}$ be a harmonic eigenfunction associated with $\lambda_2(G)$. Then for any $v \in V(G)$, we have*

$$\frac{1}{d(v)}\sum_{uv\in E(G)}(f(v)-f(u)) = \lambda_2(G)f(v).$$

Proposition 1.2 ([3]). *Let G be a connected graph of order $n \ge 2$. If G is not the complete graph, then $\lambda_2(G) \le 1$.*

Proposition 1.3 ([1]). *Let G be a connected graph. Then $\frac{\alpha(G)}{\Delta} \le \lambda_2(G) \le \frac{\alpha(G)}{\delta}$, where $\alpha(G)$ is the algebraic connectivity of G (namely, the second smallest Laplacian eigenvalue of G).*

Let v be a vertex of a graph G. Suppose that two new paths $P = vu_l\cdots u_2u_1$ and $Q = vv_k\cdots v_2v_1$ of lengths l and k ($l \ge k \ge 1$), respectively, are attached to G at v to form a new graph $G_{l,k}$, where $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_k$ are distinct. Let $G_{l+1,k-1} = G_{l,k} - v_2v_1 + u_1v_1$. We say that $G_{l+1,k-1}$ is obtained from $G_{l,k}$ by grafting an edge (see Fig. 1).

The following result due to Li et al. [8] gives the behavior on the second smallest normalized Laplacian eigenvalue when the graph is perturbed by grafting an edge. Similar result on the algebraic connectivity have been obtained by Guo in [5].

Theorem 1.4 ([8]). *Let G be a connected graph with at least two vertices, and let $G_{l,k}$, $G_{l+1,k-1}$ ($l \ge k \ge 1$) be the graphs defined above. Let $\boldsymbol{f}$ be a harmonic eigenfunction associated with $\lambda_2(G_{l,k})$. Then*

$$\lambda_2(G_{l,k}) \ge \lambda_2(G_{l+1,k-1}),$$

and the inequality is strict if either $f(v_1) \ne 0$ or $f(u_1) \ne 0$.

Moreover, let u and v be two distinct vertices of a graph G. Suppose that two new paths $P = uu_l\cdots u_2u_1$ and $Q = vv_k\cdots v_2v_1$ of lengths l and k ($l, k \ge 1$), respectively, are attached to G at u and v to form a new graph $G^2_{l,k}$, where $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_k$ are distinct. Let $G^{2+}_{l,k} = G^2_{l,k} - vv_k + u_1v_k$ and $G^{2++}_{l,k} = G^2_{l,k} - uu_l + v_1u_l$ be the graphs obtained from $G^2_{l,k}$ by grafting pendent paths Q and P, respectively. Let $\boldsymbol{f}$ be a harmonic eigenfunction associated with $\lambda_2(G^2_{l,k})$. Li et al. [7] proved that if $f(v_1)f(u_1) \ge 0$, then $\lambda_2(G^2_{l,k}) \ge \lambda(G^{2+}_{l,k})$ or $\lambda_2(G^2_{l,k}) \ge \lambda(G^{2++}_{l,k})$. Similar results on the algebraic connectivity have been obtained by Guo in [5].

Let $e = uv$ be an edge of a graph G. Let G' be the graph obtained from G by contracting the edge e into a new vertex u_e and adding a new pendent edge u_ev_e, where v_e is a new pendent vertex. We say that G' is obtained from G by separating an edge