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# Total colorings of planar graphs without chordal 6-cycles[☆]

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ABSTRACT

A total $k$-coloring of a graph $G$ is a coloring of $V(G) \cup E(G)$ using $k$ colors such that no two adjacent or incident elements receive the same color. The total chromatic number of $G$ is the smallest integer $k$ such that $G$ has a total $k$-coloring. In this paper, it is proved that if $G$ is a planar graph with maximum degree $\Delta \geq 7$ and without chordal 6-cycles, then the total chromatic number of $G$ is $\Delta + 1$.



## 1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let $G$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $\Delta$ and $\delta$) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively.

A *total k-coloring* of a graph $G$ is a coloring of $V \cup E$ using $k$ colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ has a total $k$-coloring. Clearly, $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [14] posed independently the following famous conjecture, which is known as the total coloring conjecture (TCC).

**Conjecture A.** *For any graph $G$, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$.*

This conjecture was confirmed for general graphs with $\Delta \leq 5$. For its history, readers can see [19]. For planar graphs, the only open case is $\Delta = 6$ (see [8,11]). Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, every planar graph with high maximum degree $\Delta$ has a total $(\Delta + 1)$-coloring. This result was first established in [3] for $\Delta \geq 14$, which was extended to $\Delta \geq 12$ [4], $\Delta \geq 10$ [15], and finally to $\Delta \geq 9$ [9]. Recently, Shen and Wang [12] proved that if $G$ is a planar graph with $\Delta = 8$ and $G$ contains no chordal 5-cycles or no chordal 6-cycles, then $\chi''(G) = \Delta + 1$. Wang and Wu [17] proved that if $G$ is a planar graph with $\Delta \geq 7$ and every vertex is incident with at most one triangle, then $\chi''(G) = \Delta + 1$. Wang and Wu [18] proved that if $G$ is a planar graph with $\Delta \geq 7$ and without 4-cycles, then $\chi''(G) = \Delta + 1$ (later, it is extended to $\Delta \geq 6$ by Shen and Wang [13]). Wang et al. [16] proved that if $G$ is a planar graph with $\Delta \geq 7$ and without chordal 5-cycles, then $\chi''(G) = \Delta + 1$. In this paper, we obtain that if $G$ is a planar graph with $\Delta \geq 7$ and without chordal 6-cycles, then $\chi''(G) = \Delta + 1$. To prove the result, we first establish various structural properties of $G$. Relying on these properties, we use the discharging method in the detailed proof to obtain a contradiction.

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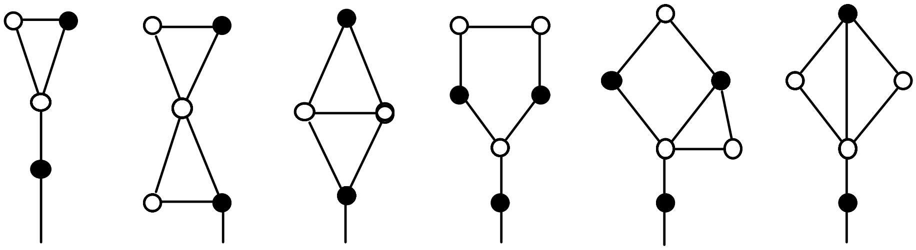

**Fig. 1.** Reducible configurations.

## 2. Main result and its proof

We will introduce some more notations and definitions here for convenience. Let $G = (V, E, F)$ be a plane graph, where $F$ is the face set of $G$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to $v$, and let $d(v) = |N(v)|$ denote the degree of $v$; and for a face $f$, the degree of a face $f$, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A *k-vertex*, a *$k^+$-vertex* or a *$k^-$-vertex* is a vertex of degree $k$, at least $k$ or at most $k$, respectively. Similarly, a *k-face* or a *$k^+$-face* is a face of degree $k$ or at least $k$, respectively. Let $n_t(v)$ be the number of $t$-vertices adjacent to a vertex $v$, and $f_k(v)$ the number of $k$-faces incident with $v$. Especially, let $f_3(v) = t$. Let $v_1, v_2, \ldots, v_d$ be neighbors of $v$ in an anticlockwise order. Let $f_i$ be face incident with $v$, $v_i$ and $v_{i+1}$, for all $i$ such that $i \in \{1, 2, \ldots, d\}$. Note that all the subscripts in the paper are taken modulo $d$. For convenience, $(d_1, d_2, \ldots, d_n)$ denotes a cycle (or a face) whose boundary vertices are of degree $d_1, d_2, \ldots, d_n$ in the anticlockwise order. Specially, $(i, j^+, k^+)$-face is a 3-face $uvw$ such that $d(u) = i \le j \le d(v) \le k \le d(w)$.

**Theorem 1.** *Let $G$ be a planar graph without chordal 6-cycles. If $\Delta \ge 7$, then $\chi''(G) = \Delta + 1$.*

**Proof.** In [12], Theorem 1 was established for $\Delta = 8$. So we assume that $\Delta = 7$. Let $G$ be a minimal counterexample to Theorem 1 in terms of the number of vertices and edges, respectively. Then every proper subgraph of $G$ has a total 8-coloring, but $G$ is not. We first show some known properties on $G$.

(a) $G$ is 2-connected and the boundary of each face in $G$ is exactly a cycle (see [5]);
(b) The subgraph $G_{27}$ of $G$ induced by all edges joining 2-vertices to 7-vertices is a forest (see [3,5]);
For any component $G_{27}$, we root it at a 7-vertex. In this case, every 2-vertex has exactly one parent and exactly one child, which are 7-vertices.
(c) $G$ contains no edge $uv$ with $\min\{d(u), d(v)\} \le \lfloor \frac{\Delta}{2} \rfloor$ and $d(u) + d(v) \le \Delta + 1$ (see [5]);
(d) $G$ contains no 3-face incident with more than one 4-vertex (see [10]);
(e) If $v$ is a 7-vertex of $G$ with $n_2(v) \ge 1$, then $n_{4^+}(v) \ge 1$ (see [6]).

**Lemma 2.** *G contains no configurations depicted in Fig. 1, where the vertices marked by • have no other neighbors in G.*

**Proof.** The proof that $G$ contains no configurations depicted in Fig. 1(1),(2),(4),(5) can be found in [7]. The proof that $G$ contains no configuration depicted in Fig. 1(3) and (6) can be found in [5,16], respectively. ∎

**Lemma 3.** *G contains no configurations depicted in Fig. 2, where the vertices marked by • have no other neighbors in G.*

**Proof.** Suppose that $G$ contains a configuration depicted in Fig. 2(1). Then $G' = G - vv_2$ has a total-8-coloring $\varphi$ with the color set $C = \{1, 2 \ldots, 8\}$ by the minimality of $G$. Erase the color on $v_2$. For a vertex $x \in V(G)$, let $C(x) = \{\varphi(xy) : y \in N(x)\}$. First, we color $vv_2$ as follows. If $|C(v_2) \cup C(v)| < 7$, then we can color $vv_2$ with a color in $C \setminus (\{\varphi(v)\} \cup C(v_2) \cup C(v))$. Otherwise, without loss of generality(WLOG), we assume that $(\varphi(vv_1), \varphi(vv_3), \varphi(vv_4), \varphi(vv_5), \varphi(v_2y), \varphi(v_2v_3), \varphi(v_2v_1), \varphi(v)) = (1, 2, 3, 4, 5, 6, 7, 8)$. If $C(v_4) \ne \{3, 5, 6, 7\}$, then we obtain a total-8-coloring of $G$ by recoloring $vv_4$ with a color in $\{5, 6, 7\} \setminus C(v_4)$, and coloring $vv_2$ with 3. Otherwise, if $\varphi(v_4) = 1$, then we exchange the colors of edges $v_3v_4$ and $vv_3$, color $vv_2$ with 2. Otherwise, we exchange the colors of edges $v_1v_2$ and $vv_1$, color $vv_4$ with 1 and color $vv_2$ with 3. Hence we obtain a total-8-coloring $\psi$ of $G$ in which $v_2$ is uncolored.

Now we begin to recolor $v_2$. Let $\alpha$ be the color on $vv_2$ and $D = C(v_2) \cup \{\alpha, 8\} \cup \{\varphi(x) : x \in N(v_2)\}$. If $|D| < 8$, then we obtain a total-8-coloring of $G$ by recoloring $v_2$ with a color in $C \setminus D$, a contradiction. Otherwise $C = D$. WLOG, we assume that $v_2y$, $v_2v_1$, $v_2v$, $v_2v_3$, $y$, $v_1$, $v$, $v_3$ is colored with 1, 2, 3, 4, 5, 6, 7, 8. First, we have $5, 6, 8 \in C(v)$, for otherwise, we recolor $vv_2$ with a color in $\{5, 6, 8\} \setminus C(v)$, and $v_2$ with 3, a contradiction. Since $d(v) = 5$ and $\{3, 5, 6, 8\} \subset C(v)$, color 2 or 4 does not appear at $v$, WLOG, $4 \notin C(v)$. If $\varphi(vv_3) \in \{5, 6\}$, then we exchange the colors of edges $v_2v_3$ and $vv_3$, recolor $v_2$ with 4. Otherwise, $\varphi(vv_3) \in \{1, 2\}$. If $\varphi(vv_3) = 1$, then we exchange the colors of edges $vv_1$ and $v_2v_1$, color $v_2$ with 2. Otherwise, we exchange the colors of edges $vv_1$ and $v_2v_1$, $v_2v_3$ and $vv_3$, and recolor $v_2$ with 4, a contradiction, too.

Suppose that $G$ contains a configuration depicted in Fig. 2(2), where $d(v) = 7$. Then $G' = G - vv_7$ has a total-8-coloring $\varphi$. Erase the colors on all black $3^-$-vertices. For a vertex $x \in V(G)$, let $C(x) = \{\varphi(xy) : y \in N(x)\}$. If $\varphi(v_7x_7) \in C(v) \cup \{\varphi(v)\}$,