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Unital designs with blocking sets

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ABSTRACT

A unital 2-(28, 4, 1) design has 28 points, each block has size 4 and every pair of points is on exactly one block. A blocking set in a design is a subset of the point set with the property that every block intersects the blocking set nontrivially but no block is contained in the blocking set. In this work, we classify the unital 2-(28, 4, 1) designs with blocking sets. We find 68,806 unitals with a blocking set. Of these, 68,484 have a trivial automorphism group.

1. Introduction and statement of results

A *finite incidence geometry* is a pair $(P, \mathcal{L})$. Here, P is a finite set whose elements are called *points*, and $\mathcal{L}$ is a set of subsets of P called *lines*, such that

1. each subset $\ell \in \mathcal{L}$ has at least two points, and
2. each pair of points is on at most one block.

A *linear space* is an incidence geometry $(P, \mathcal{L})$ such that each pair of points is on exactly one line. We refer to [3] for more details on linear spaces.

A 2-(v, k, λ) design is a pair $(P, \mathcal{L})$ such that P is a set of size v, $\mathcal{L}$ is a set of k-subsets of P, and for each pair of points in P there are exactly λ elements in $\mathcal{L}$ that contain both points. Thus, a 2-$(v, k, 1)$ design is a linear space with all lines of size k. Often, the elements of $\mathcal{L}$ in a design are called *blocks* (and the design is called a *block system*). A 2-$(v, k, 1)$ design is also known as a *Steiner* 2-*design*.

An incidence geometry $(P, \mathcal{L})$ has a *blocking set* if there exists a subset $B \subset P$ with the following properties:

1. $B \cap \ell \neq \emptyset$ for all lines $\ell \in \mathcal{L}$ and
2. $\ell \subset B$ for no line $\ell \in \mathcal{L}$.

For $n \geq 2$, a 2-$(n^3+1, n+1, 1)$ design is called a *unital design* of order n. In particular, a 2-(28, 4, 1) unital design consists of 28 points and 63 lines of size 4, so that each pair of points is on exactly one block. A *unital* is a set S of $n^3 + 1$ points in a projective plane $\pi = (P, \mathcal{L})$ of order n such that $|S \cap \ell| \in \{1, n + 1\}$ for all lines $\ell \in \mathcal{L}$.

A classical example of a unital is the Hermitian unital which is the set of absolute points of a unitary polarity in the desarguesian projective plane of order q^2, PG$(2, q^2)$. Another example is the Ree unital which is a design on $q^3 + 1$ points associated with the Ree group, see [10].

Units (i.e., unital designs that are embedded in projective planes) have received much attention. The unitals of order 3 were classified in [12]. The work [13] focuses on unitals in planes of order 16. The unitals of order 4 in the desarguesian projective plane of order 16 have been classified in [1]. The book [2] is dedicated to the study of unitals.

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Brouwer [6] was the first to show the difference between unitals and unital designs. A unital is a unital design but not conversely. The classification of *all* unital designs of order 3 seems to be beyond reach at the moment. Penttila and Royle in [12] put it this way:

The problem of constructing all 2-(28, 4, 1) *designs seems to be infeasible at this stage (though possibly not by much).*

The present work is a contribution to the classification of unital 2-(28, 4, 1) designs. Brouwer [6] found 138 unital designs of order 3. Seven more unital designs were found in [11]. In [9], unital designs with non-trivial automorphism group were classified. There are exactly 4466 such unital designs. Several unitals with trivial automorphism group were found in [5]. In the present article, we make the additional assumption that there exists a *blocking set.* Thus, we classify all unitals of order 3 admitting a blocking set. We find 68,806 such unitals, of which 68,484 have a trivial automorphism group.

The paper is organized in the following way. In Section 2, we recall some basic definition of tactical decomposition. In Section 3, the notion of a blocking set is related to that of discrepancy. In Section 4, we consider all of the possible decomposition schemes of unitals admitting a blocking set. The classification procedure is described in Section 5.

2. Tactical decomposition

In this section, we recall some basic definitions of tactical decompositions. For further reference, see [7]. In the following, we will consider an incidence geometry $\mathcal{X} = (P, \mathcal{L})$. For $p \in P$, we define

$$(p) = \{\ell \in \mathcal{L} \mid p \in \ell\}$$

the *pencil* of lines through p.

A *decomposition* of an incidence geometry $\mathcal{X} = (P, \mathcal{L})$ is a pair $(\mathcal{R}, \mathcal{C})$ of ordered partitions where $\mathcal{R} = (R_1, R_2, \ldots, R_m)$ is a partition of points and $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is a partition of blocks. For each $i = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, n$, let $r_{i,j} = |\{B \in C_j \mid p \in B\}|$ with $p \in R_i$ fixed. Let $c_{i,j} = |\{p \in R_i \mid p \in B\}|$, for $B \in C_j$ fixed.

A decomposition $(\mathcal{R}, \mathcal{C})$ of $\mathcal{X}$ is said to be *point tactical* if the number $r_{i,j}$ is independent of the choice $p \in R_i$ for each i and for each j. It is said to be *block tactical* (or *line tactical*) if for each i and for each j, the number $c_{i,j}$ is independent of the choice $B \in C_j$. A point tactical and a block tactical decomposition is simply said to be *tactical* decomposition with respect to $\mathcal{X}$.

Given a point tactical decomposition $(\mathcal{R}, \mathcal{C})$ of an incidence geometry, we define the *decomposition scheme* in the following way. These numbers are $a_i = |R_i|$ for $1 \le i \le m$, $b_j = |C_j|$ for each $1 \le j \le n$ together with the integers $r_{i,j}$ and $c_{i,j}$. In particular, if $\mathcal{R} = \{R_1, \ldots, R_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$, we use a form as in (1) to describe such a scheme.

$$\begin{array}{c|cccc} \rightarrow & b_1 & b_2 & \cdots & b_n \\ \hline a_1 & r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{array}. \tag{1}$$

Note that the horizontal arrow in the upper left corner is indicating that this scheme describes a point tactical decomposition.

In a similar way, if $(\mathcal{R}, \mathcal{C})$ is a block tactical decomposition, then the form in (2) describes the corresponding decomposition scheme. Notice that the horizontal arrow is replaced by a vertical one. This indicates that we have a block tactical decomposition.

$$\begin{array}{c|cccc} \downarrow & b_1 & b_2 & \cdots & b_n \\ \hline a_1 & c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & c_{m,1} & c_{m,2} & \cdots & c_{m,n} \end{array}. \tag{2}$$

3. Discrepancy and blocking sets

We recall the notion of discrepancy, here only applied to finite incidence geometries, see [4].

Let $(P, \mathcal{L})$ be a finite incidence structure. Let $\Pi = (A, B)$ be a non-trivial partition of the point set P with two classes (i.e., with $P = A \cup B$, with $A \cap B = \emptyset$ and with $1 < |A| < |P|$). For each line $\ell \in \mathcal{L}$, one calculates the values $|A \cap \ell|$ and $|B \cap \ell|$ and takes the positive difference. The maximum of all such numbers is

$$u(\Pi) = \max_{L} \{||A \cap \ell| - |B \cap \ell||\}.$$

The *discrepancy* of $(P, \mathcal{L})$ is the smallest value of $u(\Pi)$ when considering all non-trivial partitions of P with two classes:

$$\Delta = \min_{\Pi} u(\Pi).$$

In the present work, we are interested in 2-(28, 4, 1) designs. Since all blocks have length 4 in these designs, only the following three possibilities for the value of u exist: $4 - 0 = 4$, $3 - 1 = 2$ and $2 - 2 = 0$. It follows that the discrepancy is one of the values 4, 2 or 0. The case $\Delta = 0$ does not arise: