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Partition into almost straight trails

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ABSTRACT

Let $G = (V, E)$ be a graph that is embedded in the plane, i.e. V is a finite vertex set of points in the plane and the edge set E is represented as a set of (straight-line) segments in the plane with endpoints from V. A *trail* is a sequence $T = (e_1, \ldots, e_k)$ of pairwise distinct edges such that there are vertices $v_0, \ldots, v_k$ with $e_i = v_{i-1}v_i$ for $i \in \{1, \ldots, k\}$. Consecutive edges of a trail form an angle in the plane and with each such angle α we assign a geometrically motivated value $z(\alpha)$. The weight of T is defined as the sum of these z-values. We study the problem of partitioning the graph into trails, i.e. decomposing the edge set of the graph into a disjoint union of edge sets of trails, such that the sum of their weights is maximal. We reduce the problem to a matching problem on the circle and present an efficient matching algorithm. The problem is motivated by an application in image processing.

1. Introduction

The motivation of this paper stems from the following problem in image analysis: A noisy grey-scale image of a family of finite, almost straight and thin segments on a rectangular domain is given. The aim is to recover and to quantify these segments from the pixel-values of the image. For example, confocal laser scanning microscopic images of the actin filament, i.e. stress fibres, of cells have to be analyzed in that way.

The algorithmic solution consists of three steps: preprocessing (denoising), feature detection (ridges, branching points), and quantification. Concerning the first two steps we refer to [2,4]. We mention that there exist also other methods for feature detection like skeletonization [1] or spline interpolation [6]. But these methods do not work very well if one has many branching or crossing points of the segments. So we used a graph theoretic approach [3]: We detect points that are likely to be on a segment and check whether pairs of such points belong to one segment by computing a certain ridgeness-value for such a pair. The ridgeness-value between points P and Q describes the average filtered concavity of the interpolated pixel-values on lines that are perpendicular to the line through P and Q. If this ridgeness-value is larger than a threshold we assume that the segment between P and Q is a subsegment of an originally given segment and use this segment for later quantification.

After this feature detection we have a graph embedded in the plane whose vertices are the detected points and whose edges are the straight-line segments between points obtained by ridgeness-thresholding. In the last step, the quantification step, the most important task is the concatenation of several of these subsegments such that they altogether form an original segment which is not necessarily straight, but "almost straight".

Of course, such a concatenation can be done in many ways. But we will do this in such a way that a certain measure of straightness is maximal. Examples of randomly generated images of segments and concrete microscopic images show that this method works well.

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We mention that the degree of the vertices of the graphs that arise in our examples are relatively small, so some easier variants of matching algorithms suffice. But in this paper we solve also the case where the degrees can be arbitrarily large. We start in Section 2 with an abstract formulation of the problem, show in Section 3 that it is sufficient to solve a special variant of a matching problem on the circle, describe a fast matching algorithm in Section 4 and present examples in Section 5. We note that other variants of matching problems on the circle are studied e.g. in [5,8].

2. The problem: partition into almost straight trails

For a positive integer k, let $[k] = \{1, \dots, k\}$ and, for two integers k, ℓ with $k \le \ell$, let $[k, \ell] = \{k, k+1, \dots, \ell\}$. Let V be a finite set of points in the plane and let E be a finite set of (straight-line) segments in the plane with endpoints from V. Thus $G = (V, E)$ can be interpreted as an (undirected) graph that is embedded in the plane. As usual, we call the elements of V also *vertices* and the elements of E *edges*. A *trail* is a sequence $T = (e_1, \dots, e_k)$ of pairwise distinct edges such that there are vertices $v_0, \dots, v_k$ with $e_i = v_{i-1}v_i$ for $i \in [k]$. We denote the edge set of T by $E(T)$ and the set of *inner points* of T, i.e. $\{v_1, \dots, v_{k-1}\}$, by $\widehat{V}(T)$.

Let e, e' be two adjacent edges, i.e. e, e' have a common endpoint v. Then $\angle(e, e')$ denotes the angle that is spanned by the segments e, e' in the plane. We consider this angle as non-oriented and take a representative from the interval $[0, \pi]$.

Let an increasing function $z : [0, \pi] \to \mathbb{R}_+$ be given. In the following we consider three types of such functions, depending on a threshold θ with $\theta \in [0, \pi]$:

$$z_1(\alpha) = \begin{cases} \alpha & \text{if } \alpha \ge \theta, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

$$z_2(\alpha) = \begin{cases} \sin(\alpha/2) & \text{if } \alpha \ge \theta, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$z_3(\alpha) = \begin{cases} \max\{0, -\cos(\alpha)\} & \text{if } \alpha \ge \theta, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We use the threshold angle θ to forbid that two segments e, e' with a common endpoint v are two parts of one larger segment if $\angle(e, e') < \theta$. Therefore, we choose θ near to π in concrete applications. Moreover, in the case $\angle(e, e') \ge \theta$, the value $z(\angle(e, e'))$ may be interpreted as a quantitative measure for the combination of e and e' to a larger segment. If the two other endpoints p, p' of e and e' lie on a unit circle around v, then $z_1(\angle(e, e'))$ gives the distance between p and p' on the circle, $z_2(\angle(e, e'))$ the half of the Euclidean distance between p and p' and $z_3(\angle(e, e'))$ the length of the orthogonal projection of e' onto the line containing e if $\theta \ge \pi/2$. Of course, other examples for z are possible. We only need that z leads to the switching property introduced in Section 4.

With each trail $T = (e_1, \dots, e_k)$ we associate a weight w as follows:

$$w(T) = \sum_{i=2}^{k} z(\angle(e_{i-1}, e_i)).$$

This weight can be interpreted as a measure of "straightness" of the trail. The *Partition into Almost Straight Trails problem* (briefly *PAST*-problem) is the following:

Find a partition of G into trails $T_1, \dots, T_c$, more precisely $E = E(T_1) \dot\cup \cdots \dot\cup E(T_c)$, such that $\sum_{j=1}^{c} w(T_j)$ is maximal.

Note that we can partition the trail $T = (e_1, \dots, e_k)$ in the case $\angle(e_{i-1}, e_i) < \theta$ into two subtrails $T_1 = (e_1, \dots, e_{i-1})$ and $T_2 = (e_i, \dots, e_k)$ with $w(T) = w(T_1) + w(T_2)$. Hence we can assume w.l.o.g. that all trails $T = (e_1, \dots, e_k)$ from the partition have the property $\angle(e_{i-1}, e_i) \ge \theta$ for $i \in [2, k]$.

For illustration we study two examples with $z = z_1$. First consider a regular n-gon (see Fig. 1). Each angle has value $\frac{n-2}{n}\pi$ and hence the n-gon can be decomposed into only one trail (the whole n-gon) if $\frac{n-2}{n}\pi \ge \theta$, i.e. if $n \ge \frac{2\pi}{\pi-\theta}$, and has to be decomposed into n trails (edges), otherwise. In the first case the value of the objective function is $\frac{(n-1)(n-2)}{n}\pi$ and in the second case it is 0. Now consider a regular n-star (see Fig. 2), where n is odd. The largest angle is $\frac{n-1}{n}\pi$. So the star can be decomposed into $\frac{n-1}{2}$ trails of two edges and an additional trail of one edge if $\frac{n-1}{n}\pi \ge \theta$, i.e. if $n \ge \frac{\pi}{\pi-\theta}$, and has to be decomposed into n trails (edges), otherwise. In the first case the value of the objective function is $\frac{(n-1)^2}{2n}\pi$ and in the second case it is 0.

3. Reduction to matching problems

Let E_v be the set of edges having v as endpoint and let $\mathcal{K}_v$ be the complete graph with vertex set E_v. Here and in the following all graphs having edges of G as vertices are denoted by calligraphic letters. Let e, e' be any two vertices from $\mathcal{K}_v$. We weight the edge $\{e, e'\}$ (briefly ee') in $\mathcal{K}_v$ with

$$f(ee') = z(\angle(e, e')).$$

Thus we have a weight function f on the edge set of $\mathcal{K}_v$.