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Tilted Sperner families

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ABSTRACT

Let $\mathcal{A}$ be a family of subsets of an n-set such that $\mathcal{A}$ does not contain distinct sets A and B with $|A \setminus B| = 2|B \setminus A|$. How large can $\mathcal{A}$ be? Our aim in this note is to determine the maximum size of such an $\mathcal{A}$. This answers a question of Kalai. We also give some related results and conjectures.

1. Introduction

A set system $\mathcal{A} \subseteq \mathcal{P}[n] = \mathcal{P}(\{1, \ldots, n\})$ is said to be an *antichain* or *Sperner family* if $A \not\subset B$ for all distinct $A, B \in \mathcal{A}$. Sperner's theorem [5] says that any antichain $\mathcal{A}$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$. (See [2] for general background.)

Kalai [3] noted that the antichain condition may be restated as: $\mathcal{A}$ does not contain A and B such that, in the subcube of the n-cube spanned by A and B, they are the top and bottom points. He asked what happens if we 'tilt' this condition. For example, suppose that we instead forbid A, B such that A is $1/3$ of the way up the subcube spanned by A and B? Equivalently, $\mathcal{A}$ cannot contain two sets A and B with $|A \setminus B| = 2|B \setminus A|$.

An obvious example of such a system is any level set $[n]^{(i)} = \{A \subset [n] : |A| = i\}$. Thus we may certainly achieve size $\binom{n}{\lfloor n/2 \rfloor}$. The system $[n]^{(\lfloor n/2 \rfloor)}$ is not maximal, as we may for example add to it all sets of size $\lfloor n/4 \rfloor - 1$—but that is a rather small improvement. Kalai [3] asked if, as for Sperner families, it is still true that our family $\mathcal{A}$ must have size $o(2^n)$.

Our aim in this note is to verify this. We show that the middle layer is asymptotically best, in the sense that the maximum size of such a family is $(1 + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. We also find the exact extremal system, for n even and sufficiently large. We give similar results for any particular 'forbidden ratio' in the subcube spanned.

What happens if, instead of forbidding a particular ratio, we instead forbid an absolute distance from the bottom point? For example, for distance 1 this would correspond to the following: our set system $\mathcal{A}$ must not contain sets A and B with $|A \setminus B| = 1$. How large can $\mathcal{A}$ be?

Here the situation is rather different, as for example one cannot take an entire level. We give a construction that has size about $\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$, which is about (a constant fraction of) $1/n^{3/2}$ of the whole cube. But we are not able to show that this is optimal: the best upper bound that we are able to give is $2^n/n$. However, if we strengthen the condition to $\mathcal{A}$ not having A and B with $|A \setminus B| \le 1$ then we are able to show that the greatest family has size $\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$, up to a multiplicative constant.

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2. Forbidding a fixed ratio

In this section we consider the problem of finding the maximum size of a family $\mathcal{A}$ of subsets of $[n]$ which satisfies $p|A\setminus B| \neq q|B\setminus A|$ for all $A, B \in \mathcal{A}$ where $p:q$ is a fixed ratio. Initially we will focus on the first non-trivial case 1:2 (note that 1:1 is trivial as then the condition just forbids two sets of the same size in $\mathcal{A}$) and then at the end of the section we extend these results to any given ratio.

As mentioned in the Introduction, for the ratio 1:2 we actually obtain the extremal family when n is even and sufficiently large. This family, which we will denote by $\mathcal{B}_0$, is a union of level sets: $\mathcal{B}_0 = \cup_{i\in I}[n]^{(i)}$. Here the set I is defined as follows: $I = \{a_i : i \geq 0\} \cup \{b_i : i \geq 0\}$, where $a_0 = b_0 = \frac{n}{2}$ and a_i and b_i are defined inductively by taking $a_i = \lceil \frac{a_{i-1}}{2} \rceil - 1$ and $b_i = \lfloor \frac{b_{i-1}+n}{2} \rfloor + 1$ for all i. For example, if $n = 2^k$ then $I = \{2^{k-1}\} \cup \{2^i - 1 : 0 \leq i \leq k-1\} \cup \{2^k - 2^i + 1 : 0 \leq i \leq k-1\}$. Noting that for any sets A and B with either (i) $|A| = l$ where $l < \frac{n}{2}$ and $|B| > 2l$ or (ii) $|A| = l$ where $l > \frac{n}{2}$ and $|B| < 2l - n$ we have $|A\setminus B| \neq 2|B\setminus A|$, we see that $\mathcal{B}_0$ satisfies the required condition. Our main result is the following.

Theorem 1. *Suppose $\mathcal{A}$ is a set system on ground set $[n]$ such that $|A\setminus B| \neq 2|B\setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq (1+o(1))\binom{n}{\lfloor n/2\rfloor}$. Furthermore, if n is even and sufficiently large then $|\mathcal{A}| \leq |\mathcal{B}_0|$, with equality if and only if $\mathcal{A} = \mathcal{B}_0$.*

The main step in the proof of Theorem 1 is given by the following lemma. The proof is a Katona-type (see [4]) averaging argument.

Lemma 2. *Let $\mathcal{A}$ be a set system on $[n]$ such that $|A\setminus B| \neq 2|B\setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then*

$$\sum_{j=l}^{2l} \frac{|\mathcal{A}_j|}{\binom{n}{j}} \leq 1$$

for all $l \leq \frac{n}{3}$ and

$$\sum_{j=2k-n}^{k} \frac{|\mathcal{A}_j|}{\binom{n}{j}} \leq 1$$

for all $k \geq \frac{2n}{3}$, where $\mathcal{A}_j = \mathcal{A} \cap [n]^{(j)}$.

Proof. We only prove the first inequality, as the proof of the second is identical. Pick a random ordering of $[n]$ which we denote by $(a_1, a_2, \ldots, a_{\lceil \frac{2n}{3}\rceil}, b_1, \ldots, b_{\lfloor \frac{n}{3}\rfloor})$. Given this ordering, let $C_i = \{a_j : j \in [2i]\} \cup \{b_k : k \in [i+1, l]\}$ and let $\mathcal{C} = \{C_i : i \in [0, l]\}$. Consider the random variable $X = |\mathcal{A} \cap \mathcal{C}|$. Since each set $B \in [n]^{(i)}$ is equally likely to be C_{i-l} we have $\mathbb{P}[B \in \mathcal{C}] = \frac{1}{\binom{n}{i}}$. Thus by linearity of expectation we have

$$\mathbb{E}(X) = \sum_{i=l}^{2l} \frac{|\mathcal{A}_i|}{\binom{n}{i}}. \tag{1}$$

On the other hand, given any C_i, C_j with $i < j$ we have $|C_i \setminus C_j| = 2|C_j \setminus C_i|$ and so $\mathcal{A}$ can contain at most one of these sets. This gives $\mathbb{E}(X) \leq 1$. Together with (1) this gives the claimed inequality

$$\sum_{i=l}^{2l} \frac{|A_i|}{\binom{n}{i}} \leq 1. \quad \square$$

Proof of Theorem 1. We first show $|\mathcal{A}| \leq (1+o(1))\binom{n}{\lfloor n/2\rfloor}$. By standard estimates (see e.g. Appendix A of [1]) we have $|[n]^{(\leq \alpha n)} \cup [n]^{(\geq (1-\alpha)n)}| = o(\binom{n}{\lfloor n/2\rfloor})$ for any fixed $\alpha \in [0, \frac{1}{2})$, so it suffices to show that $|\bigcup_{i=\frac{2n}{5}}^{\frac{3n}{5}} \mathcal{A}_i| \leq \binom{n}{\frac{n}{2}}$. But this follows immediately from Lemma 2 by taking $l = \lfloor \frac{n}{3} \rfloor$.

We now prove the extremal part of the claim in Theorem 1. We first show that the maximum of $f(x) = \sum_{i=0}^{n} x_i$ subject to the inequalities

$$\sum_{j=l}^{2l} \frac{x_j}{\binom{n}{j}} \leq 1, \quad l \in \left\{0, 1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\} \tag{2}$$

and

$$\sum_{j=2k-n}^{k} \frac{x_j}{\binom{n}{j}} \leq 1, \quad k \in \left\{\left\lceil \frac{2n}{3} \right\rceil, \ldots, n\right\} \tag{3}$$