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Triple arrays and related designs

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ABSTRACT

We consider a problem posed by Donald Preece more than 30 years ago. Let S be a set of 35 distinct elements. Construct a 7×15 rectangular array A, each of whose entries is a member of S, with no symbol repeated in any row or column, with the following properties:

(P1) each symbol occurs precisely three times in the array;
(P2) any two distinct rows contain precisely five common symbols;
(P3) any two distinct columns contain precisely one common symbol;
(P4) any row and any column contain precisely three common symbols.

We shall present a solution, survey related work, and look toward further problems.

1. Introduction

We consider a problem posed by Donald Preece [18] at the Aberystwyth and Adelaide conferences, more than 30 years ago.

Let S be a set of 35 distinct elements. Construct a 7×15 rectangular array A, each of whose entries is a member of S, with no symbol repeated in any row or column, with the following properties:

(P1) each symbol occurs precisely three times in the array;
(P2) any two distinct rows contain precisely five common symbols;
(P3) any two distinct columns contain precisely one common symbol;
(P4) any row and any column contain precisely three common symbols.

If we interpret columns as treatments and symbols as blocks, then (P1) means $k = 3$, (P3) means $\lambda = 1$, and A represents a balanced incomplete block design with

$$v = 15, \qquad b = 35, \qquad r = 7, \qquad k = 3, \qquad \lambda = 1$$

(the original Steiner triple systems).

If we interpret rows as treatments and symbols as blocks, then (P1) and (P2) mean A represents a balanced incomplete block design with

$$v = 7, \qquad b = 35, \qquad r = 15, \qquad k = 3, \qquad \lambda = 5$$

(five times parameters of a Fano plane).

No solution was found for a quarter of a century when one was given in [15]. In this paper we shall describe the family of which Preece's design is a member, present a solution, and discuss some related ideas.

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2. Background

We assume the standard definitions and notations for combinatorial designs. We shall need to reference the two standard parameter relations

$$vr = bk, \tag{1}$$
$$\lambda(v-1) = r(k-1). \tag{2}$$

The usual definition of a balanced incomplete block design requires that $k < v$ ("the blocks are incomplete"), but we shall find it convenient to allow the trivial case where $k = v$.

A binary *row–column design* is a rectangular array whose entries are members of some set of *treatments*, with no repetitions in any row or column. If such a design has r rows, c columns, and v treatments which form a v-set V, then it is an $r \times c$ *binary row–column design based on* V. Such an array is also called the *Latin rectangle*, although some authors reserve this term for the case where c (or r) is equal to v. A binary row–column design is called *equireplicate* if every member of V appears the same number of times in the array; this common number is then called the *replication number* of the design.

Among binary row–column designs, perhaps the best-known is the *Latin square*, the designs with $r = c = v$, which are easily seen to exist for all values of v. Another important class is *Youden squares*. A Youden square is a $k \times v$ array based on a (v, k, λ)-*SBIBD*. Each column contains the elements of one block, ordered so that each element appears exactly once in each row. It was shown in [26] that such an ordering is always possible; that is, every symmetric balanced incomplete block design gives rise to a Youden square. (In fact, it is common for many non-isomorphic Youden squares to arise from the same *SBIBD*.)

When discussing binary row–column designs, we shall often need to refer to the set of all elements of a row, ignoring the arrangement of the elements into columns. It will be convenient to extend the usage in design theory and refer to this set as the *support* of the row, and similarly for columns.

3. Double arrays

The class of binary row–column designs that include Preece's example was defined by Agrawal [1], although a small example was discussed earlier by Potthoff [16] and another was published by Preece [17] independently of Agrawal's paper. We shall introduce these designs below under the name of *triple arrays*. We begin by introducing a more general class, *double arrays*.

Suppose $\mathcal{A}$ is an equireplicate $r \times c$ binary row–column design based on V, with replication number k, having the following properties:

(P1) any two distinct rows have the same number, λ_{rr}, of common elements;
(P2) any two distinct columns have the same number, λ_{cc}, of common elements.

Then $\mathcal{A}$ is a *double array* with parameters $v, k, \lambda_{rr}, \lambda_{cc}$, or

$$DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c).$$

Associated with any double array are two balanced incomplete block designs. To construct them, suppose the rows of a $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ are labeled as $R_1, R_2, \ldots, R_r$ and the columns are labeled as $C_1, C_2, \ldots, C_c$. Then the *row design* or $BIBD_R$ has v blocks $B_1, B_2, \ldots, B_v$, corresponding to the v elements of V: if element x appears in rows $R_a, R_b, \ldots, R_z$ then $B_x = \{a, b, \ldots, z\}$. Similarly the *column design* or $BIBD_C$ is defined using the incidence of elements in columns.

Lemma 3.1. *Suppose $\mathcal{A}$ is a $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$. Then*

(i) *the row design of $\mathcal{A}$ is a balanced incomplete block design with parameters*

$$(r, v, c, k, \lambda_{rr}).$$

(ii) *the column design of $\mathcal{A}$ is a balanced incomplete block design with parameters*

$$(c, v, r, k, \lambda_{cc}).$$

Theorem 3.2. *Any $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ satisfies*

$$vk = rc, \tag{3}$$
$$\lambda_{rr}(r-1) = c(k-1), \tag{4}$$
$$\lambda_{cc}(c-1) = r(k-1), \tag{5}$$
$$\lambda_{rr} r(r-1) = \lambda_{cc} c(c-1). \tag{6}$$

Proof. Eq. (3) follows from applying (1) to either of the designs associated with $\mathcal{A}$. Eqs. (4) and (5) are just (2), for the $BIBD_R$ and $BIBD_C$ respectively. Eq. (6) is obtained by combining (4) and (5). □